Volatility Skewness and Interest Rate Market Model

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Abstract

Observed volatility skewness in the market is an important issue to be resolved theoretically. In this paper we generalize the popular BGM model by Brace, Gatarek, and Musiela (1997) to incorporate a constant elasticity of variance. Closed form solutions for interest rate derivatives based on this model are derived. We demonstrate empirically that volatility skewness can be explained to a large extent by the generalization. The generalized model provides better fit and prediction power than the original BGM model.

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1 Introduction

Many models have been developed to study the dynamics of the term structure of interest rates. Vasicek (1977) and Cox, Ingersoll, and Ross (1985) model the instantaneous spot rate as a diffusion process and derive the equilibrium bond derivative prices. Ho and Lee (1986) and Hull and White (1990) calibrate the parameters of their spot rate processes to prevent arbitrage opportunities between bonds of different maturities. Heath, Jarrow and Morton (1992) develop another class of term structure models by taking the instantaneous forward rates as state variables. They show that the drift term of the forward rate process cannot be arbitrarily chosen, but is determined by the term structure of the volatility of forward rates under the condition of no-arbitrage. This latter model has become very popular and attractive.

However, in applying HJM-type models, there is the problem that the underlying state variables are not observable. Generally the spot rate is not Markovian, and the model may produce negative interest rates with non-zero probability for the Gaussian class. Brace, Gatarek, and Musiela (1997) (henceforth denoted as BGM), and Jamshidian (1997) introduce a new approach to no-arbitrage term structure modeling by specifying the observable forward LIBOR rate or the swap rate as the state variable. They show that by choosing an appropriate forward martingale measure rather than the usual spot martingale measure, the state variable will have the martingale representation. Thus the underlying LIBOR rate or swap rate is a lognormal diffusion process under the corresponding forward measure and the Black-Scholes type formulas are obtained for cap, floor and swaption prices. The lognormality of the state variable avoids the negative interest rate problem that plagues some of the other models. This class of models are often referred as market models since they specify the trading instruments e.g. LIBOR as state variable.

Observed volatility skewness in the market is an important issue to be resolved theoretically. For example, Kagraoka (1999)reports the volatility skew effect when he compares the HJM model versus the BGM model using Japanese money market derivatives. Blyth and Uglum (1999), and Andersen and Andreasen (1998) also address the importance of this volatility skewness issue.In figure 1 we show the implied volatility curve of the US LIBOR market. We see clearly the decreasing pattern of the volatility curve with respect to increasing strike price.

Figure 1 about here

In this paper, we generalize the market model of BGM to one incorporating constant elasticity of variance in the underlying market traded instrument. Closed form solutions for interest rate derivatives based on this model can be obtained. The volatility skewness is much reduced in our generalization.

Our model is empirically tested using the US$ LIBOR rate and cap price data from Govpx. We conduct in-sample fitting test and out-of-sample prediction test to investigate the pricing power of the CEV market model.

The paper is structured as follows. Section 2 derives the CEV-market model pricing formula. The empirical tests are described in section 3. The CEV market model is extended to the two factor case in section 4. Section 5 concludes.

2 The Model

2.1 LIBOR market

Let $B(t,T)$ denote the price of a discount bond of maturity T at time t in the interbank LIBOR market. It is assumed there is no credit risk. Define the maturity structure as $(T_0, T_1, ..., T_N)$ for $T_0 < T_1 ... < T_N = T$. The time interval between each two consequent maturity dates is equal to δ, and δ differs from product to product, but it is normally 3 months or 6 months for cap/floor and swaps. The money market account is defined as the cumulative account based on an initial investment of 1 on rolling spot interest rates $r(t)$:

$$M_t = exp\left(\int_0^t r(u)du\right).$$

The forward δ-LIBOR rates L are related to the discount bond prices as:

$$1 + \delta L(t, T_k) = \frac{B(t, T_k)}{B(t, T_{k+1})}, \qquad k = 0, ..., N-1$$

or

$$L(t, T_k) = \frac{1}{\delta}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right). \tag{1}$$

Let w be a standard brownian motion under the real probability measure $\mathbf{P}$. The bond price and forward LIBOR rate process are specified as

$$dB(t, T_k) = B(t, T_k)(a(t, T_k)dt + b(t, T_k)dw_t)$$

$$dL(t, T_k) = \mu(t, T_k, \cdot)dt + \lambda(t, T_k)L^{\beta/2}(t, T_k)dw_t$$

respectively, where $\beta > 0$ is a constant and the dot in the second formula stands for the term structure of forward rates. This CEV LIBOR rate model is more general than existing models such as BGM (1997). For example, the BGM model is obtained using $\beta = 2$ in our specification.

Under the no-arbitrage condition, there exists a unique martingale measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ with Radon-Nikodyn derivative given by

$$\eta^* = \frac{d\mathbf{P}^*}{d\mathbf{P}} = exp\left(-\int_0^T \theta(t)dw_t - \frac{1}{2}\int_0^T \theta^2(t)dt\right)$$

where $\theta(t)$ is well defined and is the market price of risk. Here we use the the money market account M_t as the numeraire.

Under the equivalent martingale measure, any dividend-adjusted asset price discounted by the money market account is a martingale. In the following, expectation is taken with respect to measure $\mathbf{P}^*$.

$$E^*\left[\left.\frac{X(T)}{M_T}\right|\mathcal{F}_t\right] = \frac{X(t)}{M_t}.$$

The notation $*$ denotes expectation under the probability measure $\mathbf{P}^*$. The dynamics of the bond price and forward δ-LIBOR rates are:

$$dB(t,T_k) = B(t,T_k)(r(t)dt + b(t,T_k)dw_t^*)$$

$$dL(t,T_k) = \mu^*(t,T_k,\cdot)dt + \lambda(t,T_k)L^{\beta/2}(t,T_k)dw_t^*$$

where

$$w_t^* = w_t + \int_0^t \theta(s)ds$$

is a standard brownian motion under probability measure $\mathbf{P}^*$.

In order to obtain the explicit solution for the LIBOR-linked security prices, we consider the forward martingale measure with respect to bonds with different maturity dates. The T_k-forward martingale measure $\mathbf{P}^{T_k}$ on $(\Omega, \mathcal{F}_{T_k})$ is defined as equivalent to $\mathbf{P}^*$, with Radon-Nikodyn derivatives given by:

$$\eta_{T_k} = \frac{d\mathbf{P}^{T_k}}{d\mathbf{P}^*} = \frac{M_{T_k}^{-1}}{E^*(M_{T_k}^{-1})} = \frac{1}{M_{T_k}B(0,T_k)}.$$

For any $0 < s < t < T_k$,

$$\begin{aligned} E_s^{T_k}\left[\frac{X(t)}{B(t,T_k)}\right] &= \frac{E_s^*\left[\frac{X(t)}{B(t,T_k)}\cdot\frac{1}{M_{T_k}B(0,T_k)}\right]}{E_s^*\left[\frac{1}{M_{T_k}B(0,T_k)}\right]} \\ &= \frac{E_s^*\left[\frac{X(t)}{M_t}\cdot\frac{M_t}{M_{T_k}}\cdot B^{-1}(t,T_k)\right]}{E_s^*[M_{T_k}^{-1}]} \\ &= \frac{E_s^*\left[\frac{X(t)}{M_t}\cdot E_t^*\left[\frac{M_t}{M_{T_k}}\right]\cdot B^{-1}(t,T_k)\right]}{M_s^{-1}B(s,T_k)} \\ &= \frac{M_s^{-1}X(s)}{M_s^{-1}B(s,T_k)} = \frac{X(s)}{B(s,T_k)}. \end{aligned} \tag{2}$$

Therefore under the T_k-forward martingale measure, the numeraire denominated security prices are martingales if we specify the bond prices with maturity of T_k as the numeraire. Thus,

$$E_s^{T_k}[L(t,T_{k-1})] = \frac{1}{\delta}\left(E_s^{T_k}\left[\frac{B(t,T_{k-1})}{B(t,T_k)}\right] - 1\right) = L(s,T_{k-1}).$$

The forward δ-LIBOR rate $L(t,T_{k-1})$ is a martingale under the T_k forward martingale measure. We have

$$dL(t,T_k) = \lambda(t,T_k)L^{\beta/2}(t,T_k)dw_t^{T_{k+1}}, \qquad 0 \le k \le N-1. \tag{3}$$

The process $w^{T_{k+1}}$ is given by

$$w_t^{T_{k+1}} = w_t^* - \int_0^t b(u, T_{k+1})du, \qquad t \in [0, T_{k+1}]$$

where w follows a standard Brownian motion under $\mathbf{P}^{T_{k+1}}$.

2.2 Cap/floor Pricing

Consider the pricing of a caplet cpl_j with notional value of 1 which is exercised at T_j and settled at T_{j+1}.If the cap rate is R, the payoff at T_{j+1} is

$$cpl_j(T_{j+1}) = \delta(L(T_j, T_j) - R)^+ .$$

From (2) we have:

$$\frac{cpl_j(t)}{B(t, T_{j+1})} = E_t^{T_{j+1}} \left[\frac{cpl_j(T_{j+1})}{B(T_{j+1}, T_{j+1})} \right]$$

so

$$cpl_j(t) = \delta B(t, T_{j+1}) E_t^{T_{j+1}}[(L(T_j, T_j) - R)^+]. \tag{4}$$

In order to use the standard method proposed by Cox (1975) and Feller (1951), we use the time changing technique in Karlin and Taylor (1981) to eliminate the time dependence of the volatility coefficient in (3). Let

$$A^j(t) = \int_0^t \lambda^2(s, T_j)ds$$

and

$$z^{j+1}(A^j(t)) = \int_0^t \lambda(s, T_j)dw_s^{T_{j+1}}.$$

Thus z^{j+1} is a one-dimension Brownian motion under the time-change $A^j(t)$. Let

$$l^j(A^j(t)) = L(t, T_j) .$$

Then (3) can be written without the time-varying coefficient term as

$$dl^j(A^j(t)) = l^j(A^j(t))^{\beta/2} dz^{j+1}(A^j(t)). \tag{5}$$

As discussed by Cox (1975), if $0 < \beta < 2$, the transition density function for SDE (5) of $l^j(\tilde{T})$ at $\tilde{T}$ conditional on $l^j(\tilde{t})$ at $\tilde{t}$, is given by

$$p(l^j(\tilde{T}), \tilde{T}; l^j(\tilde{t}), \tilde{t}) = (2-\beta) y^{1/(2-\beta)} (x w^{1-2\beta})^{1/(4-2\beta)} e^{-x-w} I_{1/(2-\beta)}(2\sqrt{xw})$$

where

$$y = \frac{2}{(2-\beta)^2(\tilde{T}-\tilde{t})}$$
$$x = y l^j(\tilde{t})^{2-\beta}$$
$$w = y l^j(\tilde{T})^{2-\beta}$$

and I is a modified Bessel function of the first kind. Hence the transaction density function for $L(T_j, T_j)$ at T_j, conditional on $L(t, T_j)$ at t is:

$$\begin{aligned} p(L(T_j, T_j), &T_j; L(t, T_j), t) \\ &= (2-\beta) y^{1/(2-\beta)} (x w^{1-2\beta})^{1/(4-2\beta)} e^{-x-w} I_{1/(2-\beta)}(2\sqrt{xw}) \end{aligned}$$

where

$$y = \frac{2}{(2-\beta)^2(A^j(T_j) - A^j(t))} \tag{6}$$
$$x = y L(t, T_j)^{2-\beta} \tag{7}$$
$$w = y L(T_j, T_j)^{2-\beta}.$$

The expectation term in the caplet pricing formula (4) can be represented as

$$\begin{aligned} E_t^{T_{j+1}} & \left[(L(T_j, T_j) - R)^+ \right] \\ &= \int_R^\infty p(L(T_j, T_j), T_j; L(t, T_j), t)(L(T_j, T_j) - R) dL(T_j, T_j) \\ &= L(t, T_j) \sum_{n=0}^\infty \frac{e^{-x} x^n G(n+1+\frac{1}{2-\beta}, yR^{2-\beta})}{\Gamma(n+1)} \\ &\quad - R \sum_{n=0}^\infty \frac{e^{-x} x^{n+\frac{1}{2-\beta}} G(n+1, yR^{2-\beta})}{\Gamma(n+1+\frac{1}{2-\beta})} \end{aligned}$$

where

$$G(m,v) = \Gamma(m)^{-1} \int_v^\infty e^u u^{m-1} du$$

is the standard complimentary gamma distribution.

Applying results in Johnson and Kotz (1972) and Schroder (1989), the solution can also be represented in terms of the non-central chi-square density functions

$$\begin{aligned} cpl_j(t) = \delta B(t, T_{j+1}) \Bigg[L(t,T_j) \left(1 - \chi^2(2yR^{2-\beta}, 2 + \frac{2}{2-\beta}, 2x) \right) \\ -R\chi^2(2x, \frac{2}{2-\beta}, 2yR^{2-\beta}) \Bigg] \quad , \qquad 0 < \beta < 2 \end{aligned}$$

where x and y are given by (6) and (7), and $\chi^2(a,b,c)$ is the cumulative non-central chi-square distribution function at a, with freedom of b and non-central parameter of c. A similar method applies when $\beta > 2$ and the caplet price is:

$$\begin{aligned} cpl_j(t) = \delta B(t, T_{j+1}) \Bigg[L(t,T_j) \left(1 - \chi^2(2x, \frac{2}{\beta-2}, 2yR^{2-\beta}) \right) \\ -R\chi^2(2yR^{2-\beta}, 2 + \frac{2}{\beta-2}, 2x) \Bigg] \quad , \qquad \beta > 2. \end{aligned}$$

See Emanuel and Macbeth (1982) for the stock options case. The cap price is easily obtained since the cap is a series of caplets with different maturity dates.

Consider the price of a floorlet with the same maturity and settlement date, and floor rate R:

$$fll_j(t) = \delta B(t, T_{j+1}) E_t^{T_{j+1}} [(R - L(T_j, T_j))^+] .$$

Hence

$$\begin{aligned} cpl_j(t) - fll_i(t) &= \delta B(t, T_{j+1}) E_t^{T_{j+1}} \left[(L(T_j,T_j) - R)^+ - (R - L(T_j,T_j))^+ \right] \\ &= \delta B(t, T_{j+1}) E_t^{T_{j+1}} (L(T_j,T_j) - R) \\ &= \delta B(t, T_{j+1}) (L(t,T_j) - R) \\ &= B(t,T_j) - (1 + \delta R) B(t, T_{j+1}) . \qquad (8) \end{aligned}$$

This is consistent with the put-call parity, which means that buying a caplet maturing at T_k and settling at T_{k+1} and also buying $1+\delta R$ units zero-coupon bond maturing at T_{k+1} is equivalent to buying the corresponding floorlet together with a zero-coupon bond maturing at T_k. This yields the floorlet pricing formula

$$
\begin{aligned}
fll_j(t) =& \delta B(t,T_j+1)\left[R\left(1-\chi^2(2x,\frac{2}{2-\beta},2yR^{2-\beta})\right)\right. \\
&\left.-L(t,T_j)\chi^2(2yR^{2-\beta},2+\frac{2}{2-\beta},2x)\right] \qquad 0<\beta<2 \\
=& \delta B(t,T_j+1)\left[R\left(1-\chi^2(2yR^{2-\beta},2+\frac{2}{\beta-2},2x)\right)\right. \\
&\left.-L(t,T_j)\chi^2(2x,\frac{2}{\beta-2},2yR^{2-\beta})\right] \qquad ,\beta>2\,.
\end{aligned}
$$

The floor price is the series summation of the corresponding floorlet prices.

2.3 Swap Market Model

In this sub-section, we discuss the swaption pricing problem. Assuming the payer's swap settled in arrears pay fixed cashflows $\delta X>0$ and receive libor interest $\delta L(T_k,T_k)$ at T_{k+1} $\quad(k=0,...,N-1)$, where the notional value is \$1. The price of this swap at time t is

$$
\sum_{k=0}^{N-1}\delta B(t,T_{k+1})(L(t,T_k)-X)\,.
$$

The forward swap rate $x(t,T_0)$ is defined as the fixed rate in this swap such that this swap is priced as 0 at time t

$$
\sum_{k=0}^{N-1}\delta B(t,T_{k+1})(L(t,T_k)-x(t,T_0))=0 \quad ,
$$

where T_0 is the first reset date of the swap and also the maturity of the swaption. Now (1) yields

$$
x(t,T_0)=\frac{B(t,T_0)-B(t,T_N)}{\sum_{k=1}^{N}B(t,T_k)}\,.
$$

In terms of the forward Libor rate, this is

$$
x(t,T_0)=\frac{\prod_{k=0}^{N-1}(1+\delta L(t,T_k))-1}{\delta\sum_{k=1}^{N}\prod_{i=k}^{N-1}(1+\delta L(t,T_i))}\,.
$$

Jamshidian (1997) points out that, according to last equation, the forward Libor rate and forward swap rate, cannot simultaneously have deterministic volatilities, and also cannot simultaneously have the martingale expression in the same equivalent probability measure. Brace, Gatarek, and Musiela (1997) and Andersen and Andreasen (1998) state that it is difficult to obtain closed form solutions for swaptions in the LIBOR-market model assuming that the forward Libor rates has deterministic volatilities. They give an approximation for European swaption prices which depends both on conditional variances and covariances of forward Libor rates of different maturities.

Another approach termed the swap market model was proposed by Jamshidian (1997). The forward swap rate is assumed to have deterministic volatility and have the martingale expression by choosing the proper numeraire for the equivalent martingale measure. Here we extend the analyses to CEV-swap market model.

Observe that

$$G(t) = \sum_{k=1}^{N} B(t, T_k),$$

where the Forward Swap Measure $\mathbf{P}^x$ is the probability measure equivalent to $\mathbf{P}^*$. We choose $G(t)$ as the numeraire. The corresponding Radon-Nikodyn derivative is

$$\eta^x = \frac{d\mathbf{P}^x}{d\mathbf{P}^*} = \frac{\sum_{k=1}^{N} B_{T_k}^{-1}}{E^*(\sum_{k=1}^{N} B_{T_k}^{-1})} = \frac{\sum_{k=1}^{N} B_{T_k}^{-1}}{G(0)}.$$

For any traded security Y and $s < t \leq T_0$,

$$\begin{aligned} E_s^x\left[\frac{Y(t)}{G(t)}\right] &= \frac{E_s^*\left[\frac{Y(t)}{G(t)} \cdot \frac{\sum_{k=1}^{N} B_{T_k}^{-1}}{G(0)}\right]}{E_s^*\left[\frac{\sum_{k=1}^{N} B_{T_k}^{-1}}{G(0)}\right]} = \frac{E_s^*\left[\frac{Y(t)}{B_t} \cdot B_t \sum_{k=1}^{N} B_{T_k}^{-1} \cdot G(t)^{-1}\right]}{E_s^*\left[\sum_{k=1}^{n} B_{T_k}^{-1}\right]} \\ &= \frac{E_s^*\left[\frac{Y(t)}{B_t} \cdot E_t^*[B_t \sum_{k=1}^{N} B_{T_k}^{-1}] \cdot G(t)^{-1}\right]}{B_s^{-1} G(s)} = \frac{Y(s)}{G(s)}. \end{aligned} \tag{9}$$

Thus

$$E_s^x[x(t, T_0)] = E_s^x\left[\frac{B(t, T_0) - B(t, T_n)}{G(t)}\right] = \frac{B(s, T_0) - B(s, T_n)}{G(s)} = x(s, T_0).$$

The forward swap rate is a martingale under the corresponding forward swap measure $\mathbf{P}^x$. Assume the forward swap rate has the Constant Elasticity of Variance

$$dx(t,T_0) = \nu(t,T_0)x^{\beta/2}(t,T_0)dw_t^x$$

where the process w^x follows a standard Brownian motion under $\mathbf{P}^x$.

Consider a European payer's swaption which gives the holder the right to enter such swap at time T_0 with fixed rate of X. The payoff of such a swaption at time T_0 is

$$swpt(T_0) = \left(\sum_{k=0}^{N-1} \delta B(T_0,T_{k+1})(x(T_0,T_0)-X)\right)^+ .$$

From (9), this swaption can be priced as

$$\begin{aligned} swpt(t) &= G(t)E_t^x\left[\frac{swpt(T_0)}{G(T_0)}\right] \\ &= \delta \sum_{k=1}^{N} B(t,T_k)\cdot E_t^x[(x(T_0,T_0)-X)^+] . \end{aligned}$$

Using the time-changing technique to eliminate the time dependence of the volatility coefficient, we apply Cox (1975), Feller (1951) and Schroder (1989)'s method and obtain the closed form solution for the swaption

$$\begin{aligned} swpt(t) &= \delta \sum_{k=1}^{N} B(t,T_k)\left[x(t,T_0)\left(1-\chi^2(2v,2+\frac{2}{2-\beta},2u)\right)\right. \\ &\quad \left. -X\chi^2(2u,\frac{2}{2-\beta},2v)\right] \qquad 0<\beta<2 \\ &= \delta \sum_{k=1}^{N} B(t,T_k)\left[x(t,T_0)\left(1-\chi^2(2u,\frac{2}{\beta-2},2v)\right)\right. \\ &\quad \left. -X\chi^2(2v,2+\frac{2}{\beta-2},2u)\right] \qquad ,\beta>2 \quad , \end{aligned}$$

where

$$\begin{aligned} u &= \frac{2X^{2-\beta}}{(2-\beta)^2(C(T_0)-C(t))} \\ v &= \frac{2x(t,T_o)^{2-\beta}}{(2-\beta)^2(C(T_0)-C(t))} \end{aligned}$$

and the time-changing variable $C(t)$ is defined as:

$$C(t) = \int_0^t \nu^2(s,T_o)ds .$$

2.4 Illustration

In Figure 2 we illustrate the volatility skew effect by the CEV market model subject to different moneyness and β. This is done by computing the CEV market model price and then implying the volatilities from Black-Scholes model based on these prices.

Figure 2 about here

The Black-Scholes model is obtained when $\beta = 2$ and when λ is specialized as a constant. This yields a flat volatility structure with respect to different strike price. When $\beta < 2$, the implied volatility curve is downward sloping with increasing strike price. Such a pattern is similar to the one observed with real market prices as shown in Figure 1.

3 Empirical test

In this section we test our CEV market model against the traditional BGM model.

3.1 Data description

Daily quote data of the US$ LIBOR cap price is used for the empirical test. Daily data for the Swapx are obtained from Govpx. The time period is from October 1, 1999 to December 23, 1999, and in each day, the cap prices with maturity from 2 years to 10 years and strike price from 4% to 10% are included. All caps are reset every 3 months. We also download the the LIBOR yield curve constructed in the money market and swap market from Govpx. We use the Govpx spread sheet of the same time to ensure that the cap price and LIBOR rate are simultaneously quoted.

3.2 Interpolation of the forward rates

Since the yield curve we have for each day contains only the yields with maturities of 3 months, 6 months, some years up to 10 years, we need to interpolate the 3-month forward libor rate of every 3-month reset for our cap pricing. In this paper, we apply the maximum smoothness forward curve method introduced by Lim and Qin (2001) to obtain the 3-month forward rates. This method is an improvement over other existing methods using linear interpolation or cubic spline methods. Essentially, we first estimate the instantaneous forward rate subject to the known points of the yield curve based on the objective of minimizing the total curvature of the forward curve. Secondly, we calculate the 3-month forward rates by the instantaneous forward curve. An illustration of this method is given by figure 3.

<u>Figure 3 about here</u>

3.3 Parameter estimation

Brace et al. (1997) use the piece-wise volatility function to empirically study their model. We also employ such function for BGM and CEV market model volatility structure, i.e. the λ is constant for each interval between two consequent cap maturity. The CEV market model has one more unobserved parameter β. We estimate the β of the CEV market model first, and then estimate the volatility functions for BGM and CEV market model by the implied parameter estimation method.

We obtain the caplet-cluster prices by calculating the deviation of two consequent caps with the same strike. For example, the 6-year cap price minus the 5-year cap price give the caplet-cluster price which is a sum of four caplet settled in $5\frac{1}{4}$, $5\frac{1}{2}$, $5\frac{3}{4}$, and 6 years with the same strike. Then we seek the parameters (λ, β) to minimize the the sum of the squares of the proportional deviations between model price of the caplet-clusters with the market prices of the same maturity for each day

$$\min_{\beta(i,j),\lambda(i,j)} \sum \left(\frac{modelprice - marketprice}{marketprice}\right)^2 .$$

We average the daily parameters. See table 1.

Table 1 about here

The β ranges from 1.06 to 1.79, which is significantly less than 2. β fluctuates considerably, and that is probably due to over-fitting in the optimization. We take the average of the βs, which is approximately 1.3, as the estimate of our β parameter.

Once β is fixed, we implement the minimization procedure once more to estimate the volatility parameters in both BGM model and CEV market model. The second and third column in Table 2 show the average of the daily volatility functions for the two models. The fourth column gives the corresponding at-the-money caplet volatility for CEV market model. This assumes a flat yield at 6.5%, and can be used to compare with the Implied BGM volatility.

Table 2 about here

3.4 In-sample fitting test

In the in-sample fitting test, we compute the CEV market model prices each day using the estimated volatility functions in the same day with the β fixed at 1.3. We divide the sample set into 3 subsets subject to different moneyness, and then divide each subset into 3 categories subject to different time-to-maturity. The average mispricing error (AME) and the root mean squared error (RMSE) within each category are calculated and compared with the same statistics by BGM model. Table 3 describes the in-sample test results.

Table 3 about here

In comparison with BGM model, the CEV market model is seen to significantly reduce the pricing error in almost every category. The overall AME and RMSE are reduced from 2.9% and 5.2% to 0.8% and 3.8% respectively.

In terms of system bias, the BGM typically under-prices the in-the-money caps, where the magnitude of AMEs of each maturity are above 6%. The CEV market model is shown to reduce this bias by half, where the overall AME is only about 3.5%. For the at-the-money category, BGM model also underprices the caps, whereas the CEV market model sometimes over-prices. The overall errors are of the similar magnitude, and the CEV market model performs much better in the short term end while the BGM is pretty good in the long end. Both CEV market model and BGM model perform well in the out-of-the money cases. Almost all AMEs for the two models are below 1%. However, the CEV market model has a smaller RMSE.

If we consider the pricing errors according to different maturities, the CEV market model has much smaller AMEs and RMSEs than the BGM model for caps of any maturity term. Especially for the mid- and long-term caps, the AMEs for CEV market model are very close to 0, while the AMEs for BGM are above 2%. The BGM model's RMSE is quite large (7.1%) for the short-term caps, while that that of CEV market model is about half.

The in-sample test indicates conclusively that our CEV market model fits market prices much better than does the BGM model.

3.5 Out-of-sample prediction test

In the out-of-sample tests, we calculate model prices based on the BGM and CEV market model with volatility functions which are estimated using data in previous day. Fro comparative purpose, the β in the CEV market model remains at 1.3. We also divide the sample set into categories subject to different maturity and moneyness, and calculate the AME and RMSE in each category. The comparison of the statistics is shown in Table 4

<u>Table 4 about here</u>

The result of the out-of-sample test is consistent with that of the in-sample test. The CEV market model significantly reduces the pricing error in almost all categories. For example, the AMEs for in-the-money caps for BGM and CEV market model are -6.5% and -3.4% respectively. That means the CEV market model reduces the underpricing bias by half at this end of the moneyness. The overall AME and RMSE for CEV market model are -0.007% and 4.6% respectively. The precision are acceptable for out-of-sample prediction, and also smaller than the corresponding measures of BGM model (-0.027%for AME and 0.59% for RMSE). Clearly, CEV market model provides a more reliable prediction of cap prices, which is exciting for both the theory and practice.

Both the in-sample and out-of-sample performance tests show that the CEV market model is a significant generalization and improvement over the BGM model. The extension in the theory is backed by the empirical and statistical testing and validation.

3.6 Hedging performance

The hedging performance of pricing models is a key consideration is the viability of any model, especially to practitioners. We shall verify empirically the hedging performance of the two models in this subsection.

Since the cap contract is a series of caplets, then to conduct a perfect delta hedge, we need to engage a series of forward or futures contracts. This is practically not feasible due to the illiquid market of the long term forward and futures contracts. It is well known that parallel shifting can explain over 80% of the yield curve movement. Therefore, another way to test hedging effectiveness is to use the swap contract which has the similar tenor.

We obtain the daily quoted swap prices from Govpx, which has tenors of 2 to 10 years. The price is in bid/ask form and we use the average as the reference price. For each contract in each day, we calculate the price changes for both models based on parallel yield curve shifts up or down by 1 basis point. We also calculate the corresponding price changes of the swap with same tenor. The hedging ratio is numerically calculated by:

$$\Delta = \frac{\text{price change of cap}}{\text{price change of swap}}$$

Based on the hedging ratio, we construct the delta-neutral portfolio by a long 1 cap contract and short Δ swap contract. The portfolio value is compared with the one calculated from the cap and swap prices of the subsequent day. In Table 5 we report the Average Hedging Ratio (AHR), Average Hedging Error (AHE) and Root Mean Square Hedging Error (RMSHE) subject to different moneyness and maturity categories.

Table 5 about here

In each category, except for the short-term deep-in-the-money caps, the CEV market model has smaller Average Hedging Errors (AHE) and smaller Root Mean Square Hedging Errors than the BGM model. By using our CEV model, the overall Average Hedging Error is reduced from 3.274 to 2.981 and the Root Mean Square Hedging Error is reduced from 1.024 to 0.987. The improvement is especially significant for the mid-, long-term and Out-of-the-Money options. The latter is the most popular category traded in the market.

Another interest finding is that the Average Hedging Ratio for the BGM model is larger than those of the CEV market model in every category, especially for the long term options. This suggests that the BGM model-based strategies may be over-hedging.

From the hedging performance tests, it is clear that in comparison with the BGM model, better hedging performance can be obtained using the CEV market model.

4 Conclusions

In this paper we show how to extend the market model of Brace, Gatarek, and Musiela (1997) and Jamshidian (1997) with the constant elasticity of variance feature to capture the strike skewness. The closed form formulas for cap/floor and swaption are derived. Using the US$ cap data, we verify empirically that the CEV market model has improved fitting and prediction power, and also better hedging performance than the famous BGM model. Our generalized interest rate market model is also able to explain the volatility skewness found in empirical data.

Table 1: CEV market model implied parameters

t	λ	β
2	0.165266849	1.794968683
2.5	0.095646443	1.326870692
3	0.067740011	1.063893248
3.5	0.093206835	1.351182236
4	0.084696448	1.265395562
4.5	0.109262189	1.405028211
5	0.104634643	1.405316431
6	0.128677731	1.626405074
7	0.085158977	1.314424535
8	0.110430368	1.533596673
9	0.111725423	1.624256614
10	0.082649487	1.276502751

Daily estimates of the parameters are obtained by minimizing the percentage pricing error in each caplet-cluster. The average of the daily parameters are then computed and shown above.

Table 2: BGM and CEV market model volatility functions

t	$\lambda(BGM)$	$\lambda(CEV)$	ATM
2.00000	0.19681	0.08035	0.20938
2.50000	0.20437	0.08360	0.21774
3.00000	0.21904	0.08934	0.23264
3.50000	0.21151	0.08615	0.22458
4.00000	0.21470	0.08774	0.22852
4.50000	0.21839	0.08865	0.23090
5.00000	0.22232	0.09028	0.23537
6.00000	0.19903	0.08083	0.21059
7.00000	0.19649	0.07988	0.20828
8.00000	0.19856	0.08051	0.20989
9.00000	0.17311	0.07048	0.18378
10.00000	0.19652	0.07978	0.20814

Daily estimates are obtained by minimizing the percentage pricing error in each caplet-cluster. The second column shows the average volatility estimated by the BGM model. The third column shows the implied volatilty parameter for CEV market model by setting $\beta = 1.3$. Column 4 shows the corresponding implied volatility for At-the-Money caplets, where the price is calculated by the CEV model.

Table 3: Cap in-sample fitting test: BGM vs CEV market model

	ITM		ATM		OTM		All	
Maturity	BGM	CEV	BGM	CEV	BGM	CEV	BGM	CEV
<4yr								
AME	-0.061	-0.035	-0.061	-0.008	-0.008	-0.012	-0.043	-0.019
RMSE	0.062	0.037	0.070	0.043	0.080	0.053	0.071	0.044
4-6 yr								
AME	-0.062	-0.028	-0.015	0.031	0.010	-0.007	-0.023	-0.002
RMSE	0.064	0.034	0.030	0.037	0.036	0.018	0.043	0.029
7-10 yr								
AME	-0.076	-0.041	-0.001	0.039	0.017	-0.005	-0.020	-0.003
RMSE	0.077	0.045	0.029	0.047	0.022	0.029	0.043	0.040
all								
AME	-0.066	-0.035	-0.026	0.020	0.006	-0.008	-0.029	-0.008
RMSE	0.068	0.039	0.043	0.042	0.046	0.033	0.052	0.038

In-sample fitting errors of BGM and CEV market model by moneyness and time-to-maturity for US$ LIBOR cap during the period October 1, 1999 to December 23, 1999 are reported. Pricing errors include Average Mispricing Error (AME) and Root Mean Square Error (RMSE) statistics. Those statistics are in proportional error terms.

Table 4: Cap out-of-sample prediction test: BGM vs CEV market model

	ITM		ATM		OTM		All	
Maturity	BGM	CEV	BGM	CEV	BGM	CEV	BGM	CEV
<4yr								
AME	-0.060	-0.035	-0.060	-0.008	-0.004	-0.009	-0.041	-0.017
RMSE	0.061	0.037	0.070	0.045	0.091	0.069	0.074	0.050
4-6 yr								
AME	-0.061	-0.028	-0.014	0.031	0.012	-0.005	-0.021	0.000
RMSE	0.064	0.034	0.034	0.041	0.051	0.036	0.050	0.037
7-10 yr								
AME	-0.075	-0.040	0.000	0.039	0.018	-0.004	-0.019	-0.002
RMSE	0.077	0.046	0.036	0.052	0.048	0.052	0.054	0.050
all								
AME	-0.065	-0.034	-0.025	0.021	0.008	-0.006	-0.027	-0.007
RMSE	0.067	0.039	0.047	0.046	0.063	0.052	0.059	0.046

Out-of-sample pricing errors of BGM and CEV market model by moneyness and time-to-maturity for US$ LIBOR cap during the period October 1, 1999 to December 23, 1999 are reported. Pricing errors include Average Mispricing Error (AME) and Root Mean Square Error (RMSE) statistics. Those statistics are in proportional error terms.

Table 5: Cap Hedging Performance Test: BGM vs CEV Market Model

	ITM		ATM		OTM		All	
Maturity	BGM	CEV	BGM	CEV	BGM	CEV	BGM	CEV
<4yr								
AHR	0.193	0.187	0.120	0.117	0.031	0.029	0.115	0.111
AHE	-0.202	-0.239	0.047	0.022	1.969	1.954	0.605	0.579
RMSHE	0.171	0.167	0.112	0.109	0.444	0.443	0.242	0.240
4-6 yr								
AHR	1.936	1.853	1.355	1.295	0.588	0.540	1.293	1.229
AHE	2.540	2.141	2.624	2.342	5.820	5.596	3.661	3.360
RMSHE	0.972	0.944	0.721	0.701	1.101	1.086	0.931	0.910
7-10 yr								
AHR	3.552	3.373	2.631	2.489	1.357	1.233	2.513	2.365
AHE	5.318	4.637	4.308	3.788	7.040	6.583	5.555	5.003
RMSHE	2.324	2.215	1.755	1.671	1.613	1.544	1.897	1.810
all								
AHR	1.893	1.804	1.369	1.300	0.659	0.601	1.307	1.235
AHE	2.552	2.180	2.326	2.051	4.943	4.711	3.274	2.981
RMSHE	1.156	1.109	0.862	0.827	1.053	1.024	1.024	0.987

Hedging performance test results of BGM and CEV market model by moneyness and time-to-maturity for US$ LIBOR cap during the period October 1, 1999 to December 23, 1999 are reported. Results include Average Hedging Ratio (AHR), Average Hedging Error (AHE) and Root Mean Square Hedging Error (RMSHE). Figures for AHE should be multiplied by 10^{-5}, and those for ASRHE should be multiplied by 10^{-3}.

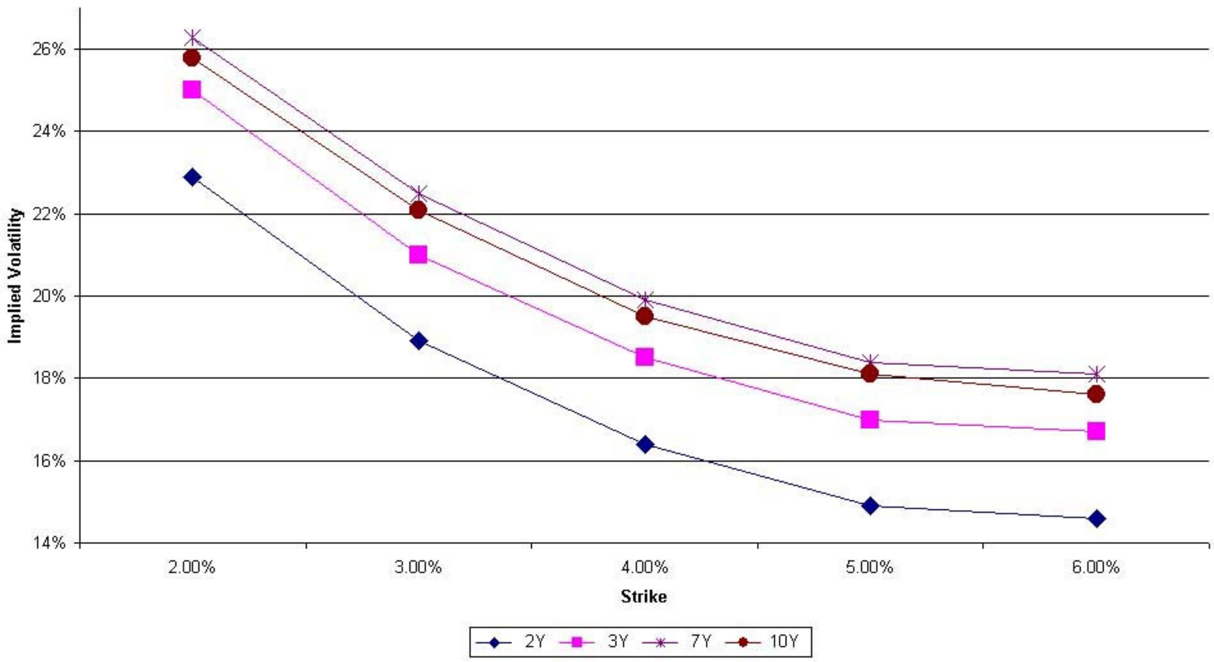

Figure 1: Caplet Implied volatilities: December 1999. Source: Reuters

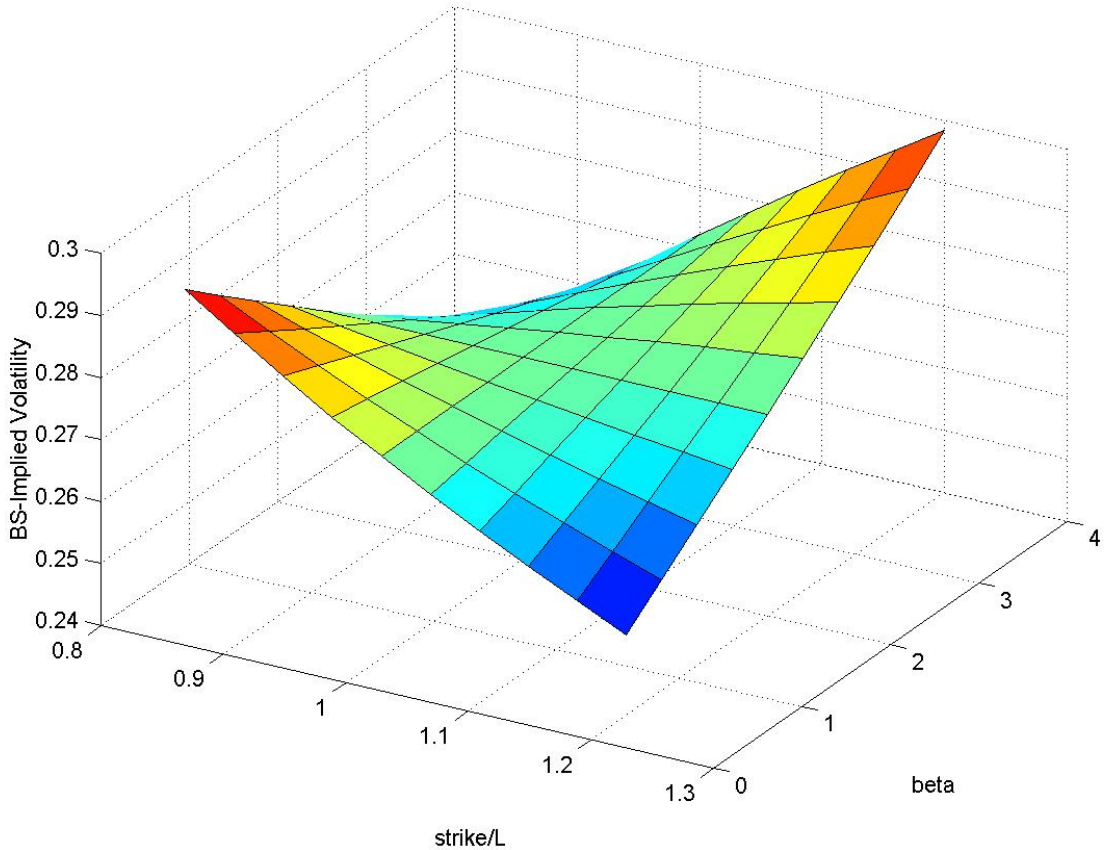

Figure 2: Volatility Skew effect by CEV market model

We fix the volatility parameters corresponding to each β in the CEV market model so that all At-the-Money caplets have the same implied volatility. The implied volatility curve is downward sloping with respect to the strike price when β is less than 2, and is upward slopping when β is greater than 2. When $\beta = 2$, volatility is flat.

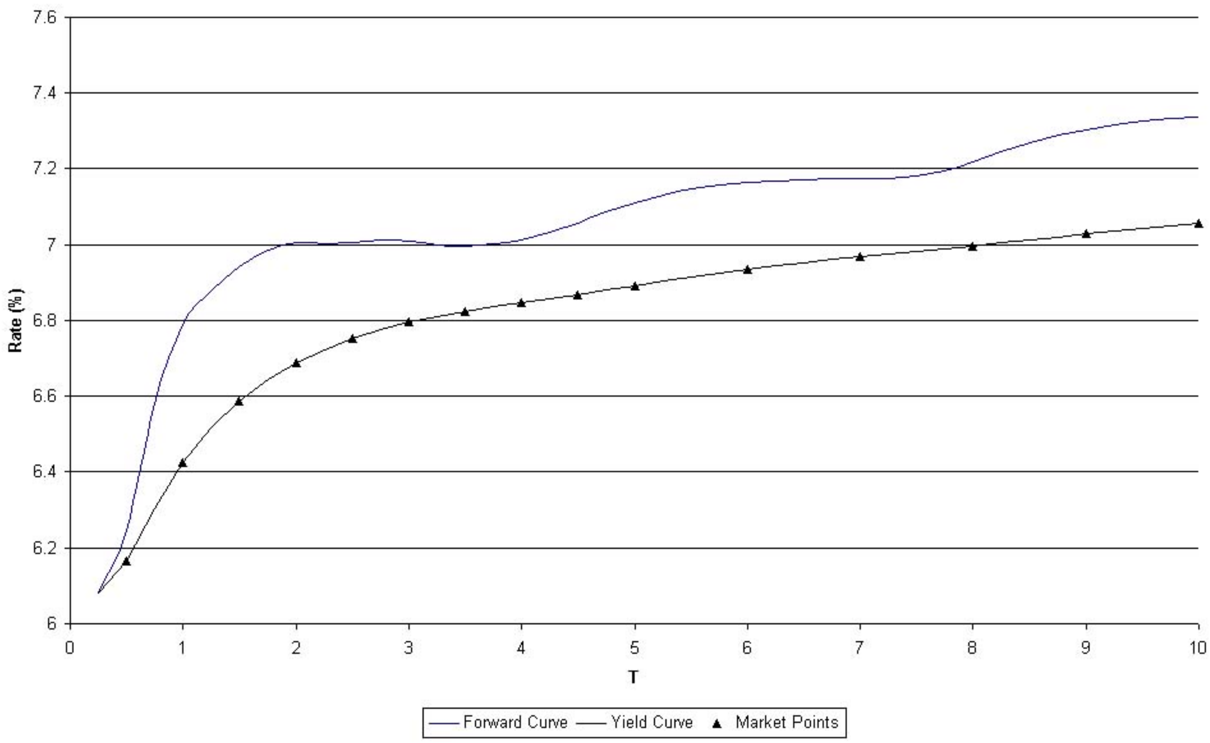

Figure 3: Maximum smoothness forward curve estimation: December 1999
We use the maximum smoothness forward curve estimation method to spline the forward rate curve and corresponding yield curve.