

A Parsimonious Monte Carlo Method for Pricing American-Style Options

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Abstract

We present a parsimonious and efficient simulation-based method for pricing American-style options. This method computes the option value by averaging the payoffs of all the scenarios at their respective optimal exercise times. At each simulation time step, a local regression is employed to estimate the option continuation values around the optimal exercise point, and the resulting function is used to reset the optimal exercise time for each scenario. This method is a significant modification of Longstaff and Schwartz' (2001) least square Monte Carlo method. We are able to dramatically reduce the scale of regression and time for computation of the option continuation value function by exploiting unique properties of the optimal exercise boundary. We demonstrate the extensive applicability of our method by pricing various vanilla and exotic American options and comparing with existing methods. Our method reduces the Monte Carlo computational time for pricing such American options to only a fraction of the existing methods. This progress in the use of Monte Carlo method is a remarkable contribution to the real-time valuation of complex path-dependent American options. The ability to reduce computational time significantly will enable continuous market-making and increase market activities for such complex options.

Keywords: Monte Carlo method, American options, exotic options.

JEL Classification: C15, C63, G13.

A Parsimonious Monte Carlo Method for Pricing American-Style Options

Monte Carlo methods have served as a flexible computational tool for the valuation of European-style derivatives. They can deal with models involving multiple variables, stochastic volatility, stochastic interest rate, and credit risk, and are well suited for the valuation of path-dependent contingent claims. Monte Carlo methods used to be thought of as a cumbersome way to price American options. The major obstacle is the lack of an accompanying process to determine whether to exercise the contract immediately or to keep it alive at each simulated point in the price-time space. This is seemingly difficult because we do not have information about the continuation value of the option if each simulated scenario is treated independently.

A number of attempts have been made to tackle this problem by using bucketing or mesh techniques to approximate the transition density function or the early exercise boundary. Examples along this line of research include Tilley (1993), Barraquand and Martineau (1995), Broadie and Glasserman (1997a,b), Broadie, Glasserman and Jain (1997), Raymar and Zwecher (1997), and Boyle et. al. (2000). Based on the stochastic mesh method, Avramidis and Hyden (1999) develop a bias-reduction technique to improve the statistical efficiency of the method. The effect of their algorithm is demonstrated in Avramidis et. al. (2000) through a parallel implementation.

Another approach developed in Li and Zhang (1996), Grant, Vora and Weeks (1997), and Garcia (2000), specifies the early exercise boundary of an American option by a finite number of parameters, and then computes the option value using maximization over the parameter space. This method is applied to Bermudan

swaptions by Andersen (1999).

Longstaff and Schwartz' (2001) least square Monte Carlo (LSM) method introduces a fundamentally different insight to the numerical determination of the early exercise boundary. The key idea is that the conditional expected payoff of the option can be estimated from the cross-sectional information using least squares. More intuitively, although a single scenario does not provide a forecast of the future, a proper aggregation of the scenarios does. The use of this technique proves successful in many cases of American-style derivatives pricing when the underlying asset follows a geometric Brownian motion or a jump-diffusion process.

Fu et. al. (2001) provides a comparative study of some of the different Monte Carlo methods. They assess the power of the methods as a function of the complexity of the product through a series of numerical experiments on the same objects. Pedersen (1999) points out that Longstaff and Schwartz's method is similar to Andersen's (1999) approach and consistent with the results obtained from low-bias stochastic mesh estimates.

Our research establishes a more straightforward, parsimonious, and efficient simulation-based method for pricing American options. A general procedure of this parsimonious Monte Carlo (PMC) method consists of a path-wise forward simulation and a backward reset of the exercise time for each scenario. The computational burden is as tractable as that in a Monte Carlo valuation of an European option. At each time step, a least squares regression is carried out within an a priori small neighborhood of the optimal exercise position. This regression is employed to estimate the function of continuation payoff in this small neighborhood, instead of over the entire domain. The result from the regression is then used either to locate the early exercise position or to evaluate the payoff of the scenarios passing through this neighborhood. For the scenarios beyond the neighborhood, their payoffs are easy to

ascertain.

This algorithm provides a significant modification to the LSM method by a parsimonious choice of a specific regression neighborhood. The specification of the regression functions can be as parsimonious as possible. We find that simple linear functions are capable of achieving a high degree of accuracy. The computational time is dramatically reduced in the implementation of our method. This advantage becomes even more significant when our method is applied to strongly path-dependent options where the continuation value depends not only on the asset price but also on one or more variables defined by the path. This will become clear in the later part of this paper on pricing exotic options. The contribution of this paper should be seen not only as a significant progress in the use of Monte Carlo method for option pricing, but also as a means toward pricing complex path-dependent American options in real time. The high-speed performance of the PMC method will enable continuous market-making and increase market activities for such options.

The rest of this paper is organized as follows. A general description of the algorithm with an illustrative example is presented in section 1. The transforms of some control variables are employed to customize the algorithm to pricing commonly traded options. Section 2 includes four subsections each studying the application of the method in the valuation of a particular American-style vanilla or exotic option. We demonstrate the performance of the PMC method by comparing its pricing accuracy and computation time with those of existing numerical methods, including the LSM method. The last section concludes the paper.

1 Formulation and Algorithm

In the Black-Scholes economy, the asset price is modeled by a geometric Brownian motion:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \quad (1)$$

under the equivalent martingale measure \mathbf{Q} on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where W_t is a Brownian motion, r and q are the continuous risk free interest rate and dividend rate respectively, and σ is the instantaneous volatility. The American option pricing problem was first formulated by McKean (1965) as a free-boundary parabolic PDE problem. The value of an American option, $P(S, t)$, with a payoff function h solves the following free boundary problem:

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S_t^2} + (r - q)S \frac{\partial P}{\partial S_t} - rP &= 0 \quad \forall (S_t, t) \in \mathcal{C}, \\ P(S_T, T) &= h(S_T), \\ P(S_t, t) &> (<) h(S_t) \quad \forall (S_t, t) \in \mathcal{C}, \text{ for put (call)} \\ P(S_t, t) &= h(S_t) \quad \forall (S_t, t) \in \mathcal{S}, \\ \frac{\partial P(S_t, t)}{\partial S_t} &= \frac{\partial h}{\partial S_t} \quad \forall (S_t, t) \in \partial\mathcal{C}, \end{aligned} \quad (2)$$

where \mathcal{C} and \mathcal{S} are the continuation region and the stopping region respectively. The optimal exercise boundary $\mathcal{B} = \{B(t) : t \in [0, T]\} = \partial\mathcal{C}$ is determined by the following condition:

$$\begin{aligned} K - B(t) &= \lim_{S_t \downarrow B(t)} P(S_t, t) \text{ for put option,} \\ \text{or } B(t) - K &= \lim_{S_t \uparrow B(t)} P(S_t, t) \text{ for call option.} \end{aligned} \quad (3)$$

The effectiveness of our Monte Carlo method for American options depends on some important properties of the optimal exercise boundary. As discussed in

van Moerbeke (1976) and Kim (1990), the optimal exercise boundary $B(t)$ for an American put has the following geometric properties:

- (i) $B(t)$ is continuously differentiable,
- (ii) $B(s) < B(t)$, if $s < t$.

Therefore, given a sufficiently fine partition $\mathcal{T} = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of the option life $[0, T]$, and provided $B(t_{n+1})$ is known, (i) and (ii) imply that $B(t_n)$ is close to the value of $B(t_{n+1})$ from below at t_n . We use this property of the boundary to locate the position of $B(t_n)$.

At any time t in the life of the option, the continuation value $P(S_t, t)$ can be expressed as the expectation of the discounted future cash flows, i.e.

$$P(S_t, t) = E^{\mathbf{Q}}[e^{-r(\tau-t)}h(S_\tau)|S_t] \quad (4)$$

where $S_\tau = \exp[(r - \sigma^2/2)(\tau - t) + \sigma(W_\tau - W_t)]$, and $\tau \in [t, T]$ is the first exit time from the continuation region \mathcal{C} . From a functional-analysis point of view, the right hand side of (4) is in fact an L^2 projection of the variable $Y = e^{-r(\tau-t)}h(S_\tau)$ onto the space $V = \{v \in L^2(\Omega) : v \text{ is } \sigma(S_t)\text{-measurable}\}$ and $\sigma(S_t)$ is the sigma algebra of S_t .¹ Therefore, if $f(S_t)$ is the actual conditional expectation, then it solves the following optimization problem:

$$\min_{v \in V} E^{\mathbf{Q}}[(v - e^{-r(\tau-t)}h(S_\tau))^2]. \quad (5)$$

The idea of estimating the conditional expectation based on the tail cash flows is attributed to Longstaff and Schwartz (2001).

A traditional Monte Carlo pricing procedure starts with a path-wise simulation of the underlying asset price. Taking the average of the payoffs over all the paths at maturity and discounting the average value to present, we obtain an estimate of

¹See Oksendal (1992) p. 59-60 for the proof.

the European-style option price

$$P_E = \frac{1}{M} e^{-rT} \sum_{m=1}^M h(S_{t_1}^m, \dots, S_{t_N}^m), \quad (6)$$

where S^m is the m -th simulated scenario, and h denotes the payoff function. Note that this formulation admits path-dependent contingent claims. We extend this algorithm to the American option pricing by expressing the value of the non path-dependent option as follows:

$$P_A = E^{\mathbf{Q}}\{e^{-r\tau} h(S_\tau)\}, \quad (7)$$

where $\tau \in [0, T]$ is the first exit time from the optimal exercise boundary \mathcal{B} . Provided that \mathcal{B} has been known in advance, we can modify (6) and construct the following simulation scheme:

$$P_A = \frac{1}{M} \sum_{m=1}^M e^{-r\tau^m} h(S_{\tau^m}), \quad (8)$$

where $\tau^m \in \mathcal{T}$ is the first exit time of the m -th scenario from the optimal exercise boundary \mathcal{B} . Unfortunately \mathcal{B} is not known in advance. So a feasible algorithm should provide a mechanism to locate τ^m . This can be achieved by performing the following procedure.

- Step 1. Generate a large number of scenarios for the underlying asset price, S_n^m , $n = 1, \dots, N$, $m = 1, \dots, M$.
- Step 2. Determine the payoff of each scenario at maturity as specified by the contract:

$$Y_N^m = h(S_N^m),$$

provided that no early exercise takes place before maturity. On the other hand, compute the value of the end point of the early exercise boundary as is given in Kim (1990) for vanilla American options:

$$B_N = \begin{cases} \max(1, \frac{r}{q})K & \text{for calls,} \\ \min(1, \frac{r}{q})K & \text{for puts,} \end{cases}$$

and initialize the exercise time:

$$\tau_N^m = T, \quad m = 1, \dots, M.$$

- Step 3. Execute the backward recursion for $n = N - 1, \dots, 1$.

Set an a priori interval $I_n \equiv [B_{t_{n+1}} - D_n, B_{t_{n+1}} + U_n]$ and select the scenarios $S_n^{m_k}, k = 1, \dots, K_n$, that pass through this interval. Then regress $(Y_{n+1}^{m_1}, \dots, Y_{n+1}^{m_{K_n}})$ on $(S_1^{m_1}, \dots, S_n^{m_{K_n}})$ by least squares and obtain an estimate of the condition expectation function $y(s)$ for the continuation value on the condition that $S_{t_n} = s, s \in I_n$ and that no earlier exercise takes place prior to t_n .

Now there are two alternatives to reset the exercise time for every scenario S^{m_k} :

- (1) Solve the equation $y(s) = h(s)$ to obtain the optimal exercise value s^* , and then take

$$\tau_n^{m_k} = \begin{cases} \tau_{n+1}^{m_k}, & S_n^{m_k} > s^* \\ t_n, & S_n^{m_k} < s^* \end{cases}$$

for a vanilla put option, for example.

- (2) Let

$$\tau_n^{m_k} = \begin{cases} \tau_{n+1}^{m_k}, & y(S_n^{m_k}) > h(S_n^{m_k}). \\ t_n, & \text{otherwise.} \end{cases}$$

The evaluation of $y(S_n^{m_k})$ and $h(S_n^{m_k})$ requires a bit more computing in comparison with (1). However, (2) is easier to implement in cases of exotic options where the functions y and h may depend on additional variables besides S . As to the scenarios beyond I_n at t_n , the readjustment of their exercise times is straightforward:

$$\tau_n^{m_k} = \begin{cases} \tau_{n+1}^{m_k}, & S_n^{m_k} > B_{t_{n+1}} + U_n, \\ t_n, & S_n^{m_k} < B_{t_{n+1}} - D_n. \end{cases}$$

Thus for all the scenarios, the payoffs are reset at t_n as follows:

$$Y_n^m = h(S_{\tau_n^m}^m), \quad m = 1, \dots, M.$$

- Step 4. Discount the payoffs to present, and average over all the scenarios at their corresponding exercise times to obtain the estimated option price

$$P_A = \frac{1}{M} \sum_{m=1}^M e^{-r\tau_1^m} h(S_{\tau_1^m}^m).$$

(Figure 1 about here)

Figure 1 presents the architecture of the regression of $Y_4^{m_k}$ on $S_3^{M_k}$ at t_3 . The regression interval is positioned about $B(t_4)$, that is the value of the optimal exercise price at t_4 . Figure 2 shows the final fixing of the first exit time from the early exercise boundary for each simulated scenario.

(Figure 2 about here)

One may note that the intuition for the above pricing process is somewhat similar to that in lattice methods used in pricing American options. Now we use a stylized example to illustrate the details of the algorithm.

Suppose we are going to value an American put option with a strike price $K = 1.10$. The risk free interest rate $r = 6\%$, and dividend rate $q = 0$. As Table I shows, we simulate 15 scenarios of three steps. Of course, in real applications, many more scenarios and a finer discretization in time are needed in order to obtain a higher accuracy.

(Table I about here)

Conditional on not exercising the option before expiration ($t=3$), the option cashflows are shown in Table II. These cash flows are identical to those that would be received if the option is European. However, at a time horizon t_n prior to maturity, the holder of the American option will decide whether to exercise the option immediately or continue to hold it. According to the definition of $B(t_n)$, this is equivalent to finding out whether $S_{t_n} > B(t_n)$. We have discussed that the position of $B(t_n)$ should be located in a small interval $[B(t_{n+1}) - D_n, B(t_{n+1}) + U_n]$ where $U_n, D_n \rightarrow 0$ as $t_{n+1} - t_n \rightarrow 0$. So we are going to set up a cross-sectional regression within a narrow interval containing the value of $B(t_{n+1})$ so as to derive an approximate function of the continuation value in this interval. Let us work backward to $t = 2$ and see how the estimation of $B(t_n)$ is carried out.

(Table II about here)

Among all the simulated prices at $t = 2$, we select the points that locate in $[1.00, 1.20]$, a neighborhood of the value $B(t_3) = K$, and denote by X the vector of their values. Let Y be the vector of the corresponding discounted payoffs and suppose the put has not been exercised by $t = 2$. Then we have the mapping from X to Y as Table III shows.

(Table III about here)

To estimate the expected reward for continuing the life of the option conditional on the prices $S_t = X$ at $t = 2$, we perform a regression of Y on a linear function of X . This specification is simple but adequate because the X here is already constrained in a small range.

The resulting linear function of the conditional expectation is $E[Y|X] = 0.6034 - 0.5045X$ and the optimal exercise price at $t = 2$ is the point that equates the continuation value and the payoff received by immediately exercising the option. Solving the linear equation

$$0.6034 - 0.5045x = 1.10 - x,$$

we obtain $x = 1.0022$, which can be regarded as an estimated value of $B(t_2)$. Now for each simulated path in Table I, if it has a price greater than 1.0022 at $t = 2$, then the option ought to be continued. Otherwise the option is exercised to obtain the intrinsic value $K - S_{t_2}$. This leads to a readjustment of the option cashflows conditional on no early exercise prior to $t = 2$. See Table IV.

(Table IV about here)

As at $t = 2$, we next locate $B(t_1)$ in a neighborhood of the value $B(t_2) = 1.002$ by means of a linear regression conditional on no occurrence of early exercise up to $t = 1$. Since the option can be exercised at $t = 2$, the payoff of each path is equal to the discounted value of the payoff received at $t = 2$ if $S_{t_2} < B(t_2)$, or else at maturity. Table V lists the prices X in the interval $[0.90, 1.10]$, and the corresponding payoffs Y if the option is continued.

(Table V about here)

The estimated conditional expectation function is $E[Y|X] = 1.1961 - 1.1009X$ and the optimal exercise price is $B(t_1) = 0.9517$. Therefore, we can make the exercise-or-hold decision at $t = 1$ and finalize the option cashflow distribution as in Table VI.

(Table VI about here)

Discounting back to $t = 0$ the payoffs received at the corresponding exercise times for all the scenarios, and taking their average, we obtain an estimated value of this American put option,

$$\begin{aligned} P_A &= \frac{d}{15} \times [(0.22 + 0.20 + 0.15 + 0.21) + d \times (0.21 + 0.12 + 0.12) \\ &\quad + d^2 \times (0.01 + 0.13 + 0.08 + 0.01 + 0.03)] \\ &= 0.0901. \end{aligned}$$

In general, the size $U_n - D_n$ of the regression interval I_n varies with t_n . It should be carefully chosen to maintain a smooth backward propagation. For example, if we are pricing an American put, an effective way to control the size of I_n is to leave the number of scenarios in the upper half of the regression interval largely unchanged. Let F be the cumulative density function of a standard normal distribution and p^* be the probability of the event $S_{t_n} > B(t_{n+1})$. Then it is straightforward that $p^* = F^{-1}(S^*)$ with

$$S^* = \frac{\ln(B_{n+1}/S_0) - (r - \sigma^2/2)t_n}{\sigma\sqrt{t_n}}. \quad (9)$$

Basically we can expect to include in the upper part of I_n , approximately $\varepsilon \times 100\%$ of all the samples $S_{t_n} > B(t_{n+1})$ by controlling U_n as indicated below:

$$U_n = S_0 e^{(r - \sigma^2/2)t_n + \sigma\sqrt{t_n}S^{**}} - S^* - B(t_{n+1}), \quad (10)$$

where $S^{**} = F^{-1}((1 - \varepsilon)p^*)$. For simplicity, we can set $D_n = U_n$.

For some deep out-of-the-money options or options on high volatility stocks, the size of the interval given in (10) may become so large that a linear base function can no longer provide a good approximation of the actual function of the continuation value. In such circumstances, it is necessary to set a termination condition at t_n when most of the sample paths deviate far away from B_{n+1} . In practice, we can choose a number $\delta \ll 1$ and stop the backward propagation when $1 - p^* < \delta$ takes

place at a certain time step t_n . In such cases, the probability of earlier exercises is negligible, and we can take the average of all the payoffs received after t_n to obtain an approximation of the option price.

2 Examples of Valuation

2.1 Valuing American Vanilla Options

Consider American put options on a stock whose price is assumed to follow the stochastic differential equation (1). In order to apply our method, we have to discretize the life of the option into a finite number of early exercise time horizons. The regression functions are specified to be linear.² We price the options with the strike price $K = 40$ and the risk free interest rate $r = 6\%$, and report in Table VII the results obtained by three different methods, viz. (a) the finite difference (FD) solution of the free-boundary problem as formulated in (2), (b) the LSM method, and (c) the PMC method. The finite difference scheme used is an implicit scheme with 52 time steps per annum and 500 steps for the stock price. Both the simulation methods generate 50,000 independent paths for the stock price with 50 discretization points per annum. We apply $\varepsilon = 0.20$ and $\delta = 0.001$ to control the size of the regression interval and the termination of the pricing procedure respectively.

([Table VII about here](#))

The three sets of numerical results are very close. There is no apparent tendency for our Monte Carlo method to undervalue or overvalue the options.

²We tried using more complex functions, but the numerical results do not improve.

Let T_{1PMC} , T_{2PMC} , T_{PMC} denote the computational times the PMC method takes for the forward simulation, the backward propagation, and the entire pricing procedure respectively, and T_{1LSM} , T_{2LSM} , T_{LSM} , the corresponding times the LMS method takes. Table VIII lists the ratios T_{2LSM}/T_{2PMC} , T_{LSM}/T_{PMC} , T_{2PMC}/T_{1PMC} , T_{2LSM}/T_{1LSM} with respect to different values of (S_0, σ, T) . The computation is executed on a PC with Intel CPU pentium III.

(Table VIII about here)

The time-saving effect is only reflected in the processes of backward propagation since the simulations in the two methods are exactly the same, i.e. $T_{1PMC} = T_{1LSM}$. In fact, if the methods are implemented on a parallel machine, then T_{2PMC} and T_{2LSM} will become dominant relative to T_{1PMC} and T_{1LSM} respectively. The first column in Table VIII shows that the discrepancy between T_{2PMC} and T_{2LSM} is significant, the latter is about three times the former. Despite the influence of T_{1PMC} and T_{1LSM} , we can still observe considerable scaling factors, T_{LSM}/T_{PMC} , between the two methods. The last two columns compare the ratios $T_{2PMC}/T_{PMC}T$ and $T_{2LSM}/T_{PMC}T$ that represent the extra computational time taken by the two methods respectively to deal with the early exercise feasibilities. Obviously, the PMC method requires much less time.

Besides pricing the option, the PMC method also provides a step-by-step backward estimation of the early exercise boundary, which is an important reference for the option holder to make an exercise decision during the life of the option. Figure 3 shows the smoothing effect on the estimated boundaries when more scenarios are simulated and incorporated in the regressions.

(Figure 3 about here)

2.2 Valuing American Barrier Options

Monte Carlo methods have been regarded as the most flexible numerical methods for pricing exotic European options. We focus on the extension of the Monte Carlo method discussed previously to the valuation of exotic American options. In this subsection, we price the Up-and-Out American barrier put options. By definition, an Up-and-Out American put option is an American put which is extinguished when the price of the underlying asset crosses a particular level from below. It is easy to apply our method to the pricing of this option. Basically what we need to do is to modify the algorithm for vanilla American put by setting to zero the payoffs of those scenarios that hit the barrier at least once before reaching, if ever, the early exercise boundary. When we perform a regression at t_n , all the simulated prices S_n in the interval I_n are used as samples even if the corresponding paths knock out before t_n .

Table IX lists the option values for $S_0 = 40$, $K = 45$, $r = 4.88\%$ and $T = 1$ obtained by the PMC method and finite-difference methods. The finite difference (FD) scheme used here is explicit with 4,000 time steps per annum and 100 steps for the stock price. The PMC method simulates 50,000 scenarios with 50 nodes per annum. Again the control parameters ε and δ are equal to 0.20 and 0.001 respectively.

(Table IX about here)

Taking the results of the FD method as a benchmark, we observe that the two methods come up with close results for the cases of low volatility and large distance between the spot and the barrier.³

³Figlewski and Gao (1999) discusses the problem of accuracy for lattice and simulation methods when volatility is significantly large or when the spot price is close to the barrier.

2.3 Valuing American Asian Options

An American Asian option is a path-dependent American option whose payoff is based on the average of the underlying asset price. The pricing of Asian options with early exercise provision poses a challenge in computational finance. To use finite difference methods, one has to transform the path-dependency to a Markovian structure. This results in higher dimensional numerical schemes and thus a dramatic increase in computing time. As for the tree methods, the primary difficulty is that the distribution of the average price conditional on the spot price is hard to characterize based on a lattice structure. Hull and White (1993) have reported an approach to compute an upper bound on the price. Chalasani et. al. (1999) develop an improved algorithm to estimate the upper and lower bounds.

Although the simulation of path-dependent variables is not difficult, the estimation of the optimal exercise rule is an impediment as in the pricing of American vanilla options. The LSM method circumvents the difficulty at a costly price: a total of eight basis functions are used in their regressions. Otherwise the estimated conditional expectation function might not be able to capture the feature over the entire in-the-money domain. In addition, the evaluation of the function in this domain also requires a tremendous amount of computing. In contrast, our method only needs to perform the regression in a neighborhood of the early exercise boundary, which is now a two-dimensional manifold. Therefore, the regression scale will shrink and the evaluation of the estimated function of the continuation value can be avoided or significantly reduced. We can still employ parsimonious linear functions as the base functions for the regressions.

For an Asian option, the function of continuation value at time t is no longer a

single-variable function of S_t . It also depends on the average price

$$A_t = \frac{1}{t} \int_0^t S_s ds,$$

or discretely,

$$A_{t_n} = \frac{1}{n+1} \sum_{i=0}^n S_{t_i}.$$

Accordingly, the optimal exercise boundary is not simply a curve in the two-dimensional asset-time plane, but a surface in a space of three dimensions. At any time step $t_n \in \mathcal{T}$, we employ a regression on a linear base function

$$f(S_{t_n}, A_{t_n}) = \alpha_{1,n} S_{t_n} + \alpha_{2,n} A_{t_n} + \alpha_{3,n}$$

within a small region of (S_{t_n}, A_{t_n}) selected as follows:

$$-D_n < \alpha_{1,n+1} S_n + \alpha_{2,n+1} A_n + \alpha_{3,n+1} - h(A_n) < U_n,$$

provided that $\alpha_{1,n+1}, \alpha_{2,n+1}, \alpha_{3,n+1}$ have been obtained in the regression at the next time step. As in the case of vanilla American options, U_n and D_n are two small positive numbers such that the regression estimates are robust.

We consider American arithmetic average call options on a stock with the spot price $S_0 = 50$, the volatility $\sigma = 0.30$, and the risk free interest rate $r = 10\%$. Table X reports values obtained by the Hull-White (HW)(1993), PMC and LSM methods. The simulations comprise 50,000 scenarios of 40 steps for each option. The control parameters $U_n = D_n = 4.0 + (K - S_0)/5.0$ and $\delta = 0.001$ are used in the PMC method. The implementation of the HW methods employs a grid size $h = 0.005$ and $n = 40$.

([Table X about here](#))

The PMC and HW results are very close to each other for many cases of in-the-money, at-the-money and out-of-the-money options. The estimated function of continuation value near the optimal exercise boundary changes smoothly with time. Table XI presents the outcomes for $K = 40$ and $T = 1$. This provides support for our method to readjust the position of regression domains in a similar pattern as in the valuation of vanilla American options, making use of the smoothness of the early exercise boundary.

(Table XI about here)

As Longstaff and Schwartz (2000) suggest, we use eight base functions for the regressions in the LSM method. To present a fair comparison of computational times, we set the eight base functions to be the simplest polynomials, namely, $1, S_n, A_n, S_n^2, A_n^2, S_n A_n, S_n^2 A_n$ and $S_n A_n^2$, and employ straightforward the Gaussian elimination to solve the resulting simultaneous linear equations of the regressions.⁴ Table XII compares the time costs of the LSM and PMC methods. We observe a more significant time-saving effect than in the case of American vanilla options. The first two column show that T_{2LSM} and T_{LSM} now multiply T_{2PMC} and T_{PMC} respectively by larger factors compared to those in table 8. Due to the computational complexity caused by the path dependency of Asian options, the LSM method has to spend a dominant portion of time, T_{2LSM} , in the backward propagation, while the PMC can still restrict T_{1PMC} to only a small fraction of T_{PMC} .

(Table XII about here)

⁴It should be noted that LSM could use more complicated base functions. However, more computational time would then be needed is needed in order to achieve a higher degree of accuracy.

2.4 Valuing American Lookback Options

In this subsection, we study the performance of our method in pricing American lookback options. The options priced here are floating strike American lookback calls, whose strike corresponds to the minimum price m_t realized by the underlying asset up to the exercise time t . The volatility of the underlying asset is $\sigma = 20\%$, the risk free interest rate is $r = 6\%$, the continuous dividend yield is $q = 3\%$, and the realized minimum price is $m_0 = 75$. We apply the PMC method to price one-year options with respect to different spot prices S_0 of the underlying asset. 50,000 sample paths of 256 steps are simulated for each option, and the parameters $U_n = D_n = 1.0$ and $\delta = 0.001$ are used to control the backward propagation process. The results are compared with those produced by the binomial schemes of Babbs (1992) with 256 time steps in Table XIII. The performances of the recursive integral method, the Crank-Nicolson scheme and compact scheme with respect to the same set of parameters are reported in Yu et. al. (1997). A linear programming formulation of the valuation of exotic American options including lookbacks is proposed in Dempster et. al. (1998).

We use bivariate linear functions as regression functions and find that they perform very well as in the pricing of American Asian options. Table XIII shows that the PMC results are almost identical to those produced by Babbs' binomial trees for a sequence of the spot price of the underlying asset.

(Table XIII about here)

3 Conclusions

This paper presents a parsimonious, efficient and flexible Monte Carlo method for pricing American-style options. Compared with other simulation-based methods, this method is intuitively very similar to the Monte Carlo methods used in the valuation of European options, but yet affords an efficient mechanism to estimate the early exercise boundary. The algorithm shares some similarity with lattice methods in the way it incorporates the optimal exercise principle in the backward propagation. Our method is a significant modification of Longstaff and Schwartz' (2000) global regression method to estimate the continuation values. It dramatically reduces the computation time by deminishing the scale of regressions and the computing burden to estimate the continuation value functions. The method presented here can serve as a unified numerical means to estimate the values of both vanilla and exotic types of American options. The paper has demonstrated its extensive applicability and remarkable efficiency through the numerical pricing of American vanilla, barrier, lookback and Asian options. More complicated products with early exercise provision can be priced by the use of our method. The success of the PMC method is not only a contribution to the literature on numerical option pricing, but also an improvement on the computational efficiency of market-making and other real-time practices in derivative markets.

Table I
Sample Paths of the Underlying Asset Price

This table presents 15 presumed scenarios of three steps for the price of the underlying asset starting from $S_0 = 1.00$.

Path	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t = 0$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$t = 1$	0.88	0.90	0.96	1.16	1.14	1.02	0.95	1.11	0.89	1.07	1.18	1.05	1.11	0.99	1.18
$t = 2$	1.11	0.87	0.89	1.23	1.17	1.03	0.97	0.98	0.85	1.21	1.15	1.08	0.98	1.04	1.12
$t = 3$	1.00	0.82	0.81	1.35	1.09	0.97	1.02	0.85	0.82	1.34	1.28	1.02	0.92	1.09	1.07

Table II
Option Cashflows at Maturity

This table shows the payoffs at maturity for the scenarios in Table I conditional on no early exercise of the option before maturity.

Path	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t = 3$	0.10	0.28	0.29	0.00	0.01	0.13	0.08	0.25	0.28	0.00	0.00	0.08	0.18	0.01	0.03

Table III
Regression at $t = 2$

The values of X in this table are the samples of S_2 within I_2 , and the values of Y are the payoffs for the corresponding scenarios, where $d = \exp(-.06)$, is the one-step discount factor. A regression of Y on X is performed to estimate the continuation value function on I_2 .

Path	1	5	6	11	12	14	15
X	1.11	1.17	1.03	1.15	1.08	1.04	1.12
Y	$0.10*d$	$0.01*d$	$0.13*d$	0.00	$0.08*d$	$0.01*d$	$0.03*d$

Table IV
Readjustment of Option Cashflows

This table shows the readjustment of the option cashflows at $t = 2, 3$ conditional on no early exercise at $t = 1$.

Path	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t = 2$	-	0.23	0.21	-	-	-	0.13	0.12	0.25	-	-	-	0.12	-	-
$t = 3$	0.10	-	-	0.00	0.01	0.13	-	-	-	0.00	0.00	0.08	-	0.01	0.03

Table V
Regression at $t = 1$

A regression of the payoffs Y on the prices of the underlying asset X is performed to estimate the continuation value function on the interval $I_1 = [0.90, 1.10]$.

Path	2	3	6	7	10	12	14
X	0.90	0.96	1.02	0.95	1.07	1.05	0.99
Y	$0.23*d$	$0.21*d$	$0.13*d^2$	$0.13*d$	0.00	$0.08*d^2$	$0.01*d^2$

Table VI
Final Option Cashflows

This table shows the final option cashflows following the optimal exercise strategy determined by the backward propagation procedure.

Path	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t = 1$	0.22	0.20	-	-	-	-	0.15	-	0.21	-	-	-	-	-	-
$t = 2$	-	-	0.21	-	-	-	-	0.12	-	-	-	-	0.12	-	-
$t = 3$	-	-	-	0.00	0.01	0.13	-	-	-	0.00	0.00	0.08	-	0.01	0.03

Table VII
Prices of Vanilla American Puts

This table reports the American put values with the strike $K = 40$, the risk free interest rate $r = 6\%$, and different combinations of the spot of the underlying S_0 , the volatility σ and the time to maturity T . The results are obtained by (a) an implicit scheme of the finite difference (FD) method, (b) the LSM method, (c) the PMC method. The FD method takes 52 steps per annum and 500 steps for the stock price. The two simulation methods both generate 50,000 sample paths of 50 time steps per annum. The control parameters for the PMC method are $\varepsilon = 0.20$ and $\delta = 0.001$.

S	σ	T	FD	LSM	PMC
36	0.20	1	4.464	4.463	4.480
36	0.20	2	4.829	4.819	4.844
36	0.40	1	7.078	7.107	7.134
36	0.40	2	8.488	8.498	8.489
38	0.20	1	3.237	3.247	3.255
38	0.20	2	3.734	3.730	3.714
38	0.40	1	6.123	6.138	6.151
38	0.40	2	7.649	7.669	7.644
40	0.20	1	2.301	2.310	2.302
40	0.20	2	2.875	2.868	2.845
40	0.40	1	5.288	5.310	5.297
40	0.40	2	6.898	6.920	6.895
42	0.20	1	1.606	1.621	1.596
42	0.20	2	2.203	2.196	2.174
42	0.40	1	4.559	4.594	4.553
42	0.40	2	6.225	6.249	6.177
44	0.20	1	1.101	1.121	1.093
44	0.20	2	1.682	1.683	1.647
44	0.40	1	3.925	3.954	3.918
44	0.40	2	5.623	5.640	5.565

Table VIII
Comparison of Computational Times for Vanilla American
Puts

This table compares the efficiency in terms of time ratios between the LSM and PMC methods in pricing American vanilla puts with the same pricing conditions as in Table VII. The notations T_{1PMC} , T_{2PMC} , T_{PMC} denote the computational times the PMC method takes for the forward simulation, the backward propagation, and the entire pricing procedure respectively, and T_{1LSM} , T_{2LSM} , T_{LSM} denote the corresponding times the LMS method takes. The computation is executed on a PC with Intel CPU pentium III.

S_0	σ	T	T_{2LSM}/T_{2PMC}	T_{LSM}/T_{PMC}	T_{2PMC}/T_{PMC}	T_{2LSM}/T_{LSM}
36	0.20	1	6.61	1.69	0.14	0.93
36	0.20	2	6.45	1.62	0.13	0.83
36	0.40	1	7.03	1.64	0.12	0.84
36	0.40	2	6.73	1.61	0.12	0.81
38	0.20	1	6.23	1.57	0.12	0.77
38	0.20	2	5.82	1.52	0.12	0.70
38	0.40	1	6.37	1.56	0.12	0.74
38	0.40	2	6.83	1.57	0.11	0.73
40	0.20	1	5.41	1.42	0.10	0.57
40	0.20	2	5.17	1.40	0.11	0.55
40	0.40	1	6.12	1.48	0.10	0.64
40	0.40	2	6.09	1.49	0.11	0.65
42	0.20	1	3.97	1.27	0.10	0.39
42	0.20	2	4.09	1.28	0.10	0.41
42	0.40	1	5.71	1.41	0.10	0.54
42	0.40	2	5.73	1.44	0.10	0.58
44	0.20	1	2.90	1.16	0.09	0.27
44	0.20	2	3.27	1.20	0.10	0.32
44	0.40	1	4.85	1.33	0.10	0.46
44	0.40	2	5.27	1.38	0.10	0.52

Table IX
Prices of Up-and-Out American Puts

This table reports the up-and-out American put values with the spot of the underlying $S_0 = 40$, the strike $K = 45$, the risk free interest rate $r = 4.88\%$, the time to maturity $T = 1$, and different combinations of the volatility σ and the position of the knock-out barrier H . The results are computed by (a) an explicit scheme of the finite-difference (FD) method with 4000 time steps per annum and 100 steps for the stock price, (b) the PMC method with 50,000 scenarios of 40 time steps, and the control parameters $\varepsilon = 0.20$ and $\delta = 0.001$.

σ	0.10		0.20		0.30		0.40		0.50	
Method	MC	FD	MC	FD	MC	FD	MC	FD	MC	FD
$H = 50$	5.0000	5.0000	5.4340	5.3851	6.3294	6.1451	7.1233	6.7053	7.7736	7.1056
$H = 55$	5.0000	5.0000	5.5337	5.5261	6.7903	6.7286	7.9390	7.7511	8.8992	8.5329
$H = 60$	5.0000	5.0000	5.5391	5.5377	6.8946	6.8847	8.2676	8.1939	9.4574	9.2895
$H = 65$	5.0000	5.0000	5.5394	5.5393	6.9197	6.9199	8.3860	8.3650	9.7730	9.6709

Table X
Prices of American Asian Options

This table compares the numerical results from the HW, PMC and LSM methods for American arithmetic average calls with the spot of the underlying $S_0 = 50$, the volatility $\sigma = 0.30$, and the risk free interest rate $r = 10\%$, and different combinations of the strike K and the time to maturity T . The simulations comprise 50,000 scenarios of 40 steps for each option. The control parameters for the PMC method are $U_n = D_n = 4.0 + (K - S_0)/5.0$ and $\delta = 0.001$. The HW method employs a grid size $h = 0.005$ and $n = 40$.

K	T	HW	PMC	LSM
40	0.5	12.115	12.117	11.503
40	1.0	13.153	13.157	13.082
40	1.5	13.988	13.988	13.813
40	2.0	14.713	14.715	14.656
45	0.5	7.261	7.267	7.128
45	1.0	8.551	8.533	8.247
45	1.5	9.652	9.622	9.545
45	2.0	10.623	10.582	10.504
50	0.5	3.275	3.239	3.157
50	1.0	4.892	4.856	4.771
50	1.5	6.199	6.162	6.014
50	2.0	6.326	7.285	5.923
55	0.5	1.152	1.136	1.031
55	1.0	2.536	2.509	2.490
55	1.5	3.771	3.733	3.708
55	2.0	4.866	4.855	4.534
60	0.5	0.322	0.323	0.326
60	1.0	1.208	1.201	1.194
60	1.5	2.194	2.181	2.155
60	2.0	3.171	3.153	2.689

Table XI
Change in Continuation Values near Optimal Exercise
Boundary

This table shows the smooth change of estimated coefficients α_1 , α_2 , and α_3 of the linear base functions for the option continuation values. The results in this table correspond to the case for the strike $K = 40$ and the time to maturity $T = 1$. The other variables are the same as in Table X.

T	α_1	α_2	α_3
1.000	0.000	1.000	-40.000
0.950	0.041	0.958	-39.903
0.900	0.071	0.929	-39.821
0.850	0.095	0.904	-39.655
0.800	0.117	0.880	-39.448
0.750	0.134	0.865	-39.444
0.700	0.152	0.847	-39.311
0.650	0.170	0.830	-39.229
0.600	0.187	0.814	-39.189
0.550	0.202	0.801	-39.179
0.500	0.227	0.777	-39.110
0.450	0.250	0.756	-39.043
0.400	0.274	0.730	-38.886
0.350	0.298	0.714	-39.106
0.300	0.339	0.667	-38.628
0.250	0.387	0.618	-38.397
0.200	0.444	0.562	-38.224
0.150	0.497	0.510	-38.115
0.100	0.625	0.371	-37.216
0.050	0.767	0.238	-37.381

Table XII
Comparison of Computational Times for American Asian
Options

This table compares the efficiency in terms of time ratios between the LSM and the PMC methods in pricing American arithmetic average calls with the same pricing conditions as in Table X. T_{1PMC} , T_{2PMC} , T_{PMC} and T_{1LSM} , T_{2LSM} , T_{LSM} , follow the same definitions as in the description of Table X, and the computation is exercised on the same PC.

S_0	T	T_{2LSM}/T_{2PMC}	T_{LSM}/T_{PMC}	T_{2PMC}/T_{PMC}	T_{2LSM}/T_{LSM}
40	0.5	9.43	2.99	0.31	2.91
40	1.0	9.81	2.95	0.28	2.79
40	1.5	10.52	2.96	0.26	2.73
40	2.0	10.93	2.96	0.25	2.69
45	0.5	8.58	2.77	0.30	2.62
45	1.0	9.22	2.73	0.27	2.46
45	1.5	9.65	2.71	0.25	2.37
45	2.0	10.14	2.70	0.23	2.32
50	0.5	6.69	2.03	0.22	1.48
50	1.0	7.31	2.09	0.21	1.52
50	1.5	8.01	2.14	0.19	1.56
50	2.0	8.29	2.17	0.19	1.58
55	0.5	3.90	1.26	0.10	0.39
55	1.0	4.99	1.44	0.12	0.62
55	1.5	5.54	1.55	0.14	0.77
55	2.0	6.17	1.64	0.14	0.88
60	0.5	2.06	1.06	0.06	0.11
60	1.0	3.22	1.16	0.08	0.26
60	1.5	4.08	1.26	0.09	0.38
60	2.0	4.71	1.35	0.10	0.49

Table XIII

Prices of Floating Strike American Lookbacks

This table provides the values of floating strike American lookback options produced by the PMC method and Babbs binomial scheme with the interest rate $r = 6\%$, the continuous dividend rate $q = 3\%$, the realized minimum price $m_0 = 75$, the time to maturity $T = 1$, and different spot prices S_0 . The PMC method simulates 50,000 sample paths and uses the control parameters $U_n = D_n = 1.0$ and $\delta = 0.001$. The number of time steps is 256 for both methods.

S_0	75	80	85	90	95	100
Babbs	11.49	13.00	15.58	18.95	22.87	27.16
MC	11.47	13.01	15.53	18.96	22.87	27.15

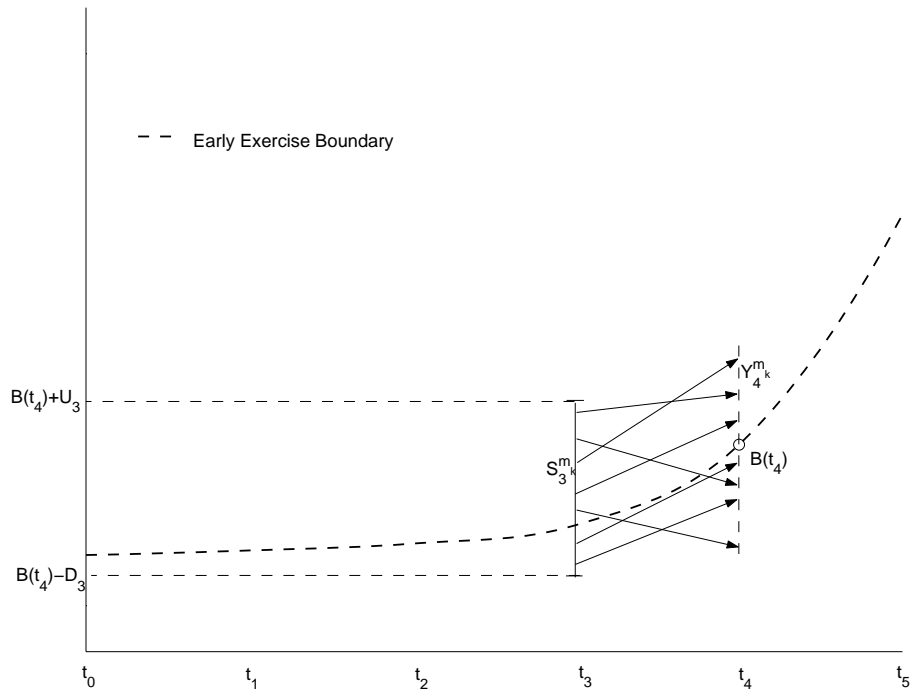


Figure 1. An Illustration of the Algorithm for the American Put (I). This figure depicts the architecture of the regression of $Y_4^{m_k}$ on $S_3^{M_k}$ at t_3 . The regression interval is positioned about the point $(t_3, B(t_4))$, where $B(t_4)$ is critical price of the underlying asset at t_4 .

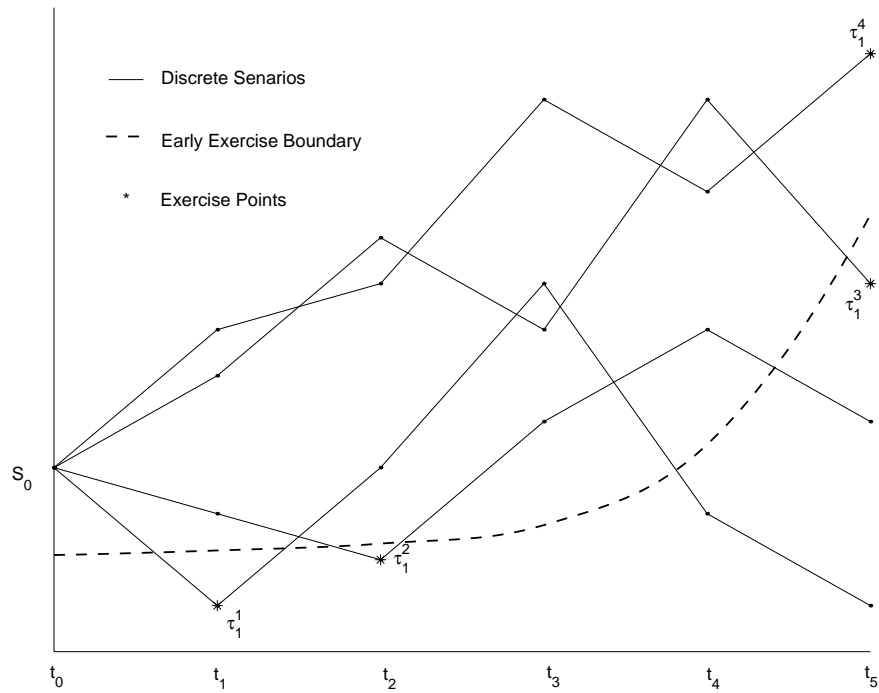


Figure 2. An Illustration of the Algorithm for the American Put (II). The PMC methods computes an option value by averaging the discounted payoffs received at the corresponding exercise times for all the simulated scenarios. This figure shows that the fixing of the exercise times for the scenarios is equivalent to the determination of their first exit times from the optimal exercise boundary.

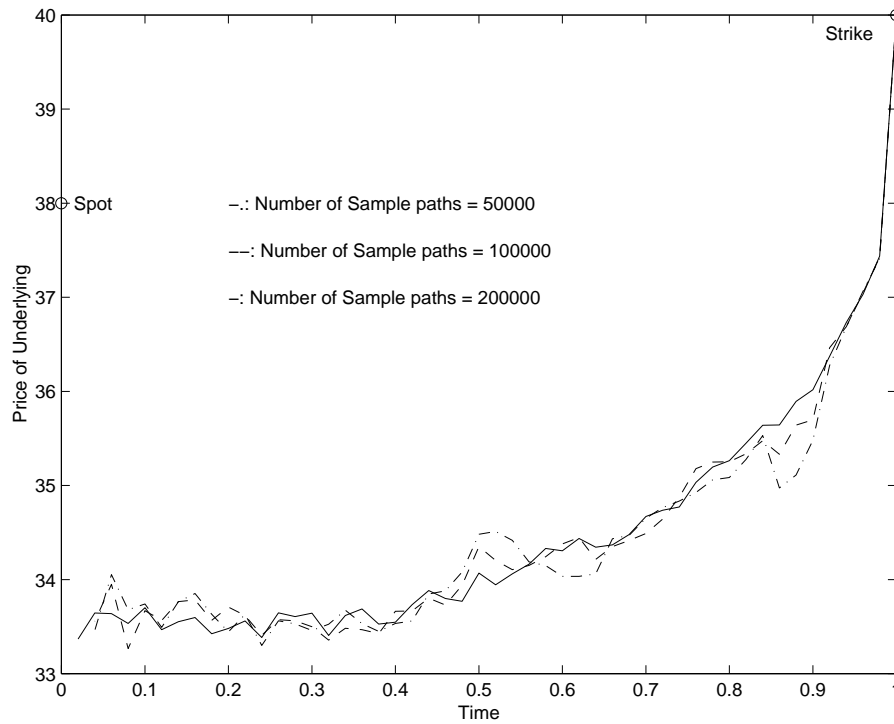


Figure 3. Smoothness of Estimated Early Exercise Boundaries. The backward propagation process in the PMC method provides an estimation of the early exercise boundaries for American put options. This table displays the smoothing effect on the estimated boundaries as the number of simulated scenarios increases.

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