# Pricing American Options with Stochastic Volatility: Evidence from S&P 500 Futures Options

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**Abstract**

This paper is the first attempt to empirically test a numerical solution to price American options under stochastic volatility. The model allows for mean reverting stochastic volatility process with non-zero risk premium for the volatility risk and correlation with the underlying process. A general solution of risk-neutral probabilities and price movements is derived, which avoids the common negative probability problem in numerical option pricing with stochastic volatility. The empirical test shows clear evidences supporting the occurrence of stochastic volatility. The stochastic volatility model outperforms the constant volatility model by producing smaller bias and better goodness-of-fit in both the in-sample test and the out-of-sample test. It not only eliminates systematic moneyness bias produced by the constant volatility model, but also has better prediction power. In addition, both models perform well in the dynamic intraday hedging test. However, the constant volatility model seems to have a slightly better hedging effectiveness. The profitability test shows that the stochastic volatility is able to capture statistically significant profits while the constant volatility model produces losses.

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## 1 Introduction

Option pricing is one of the most exciting areas in modern financial research. Ever since the famous Black and Scholes (1973)[6] paper, there are many extensions and modifications to the option pricing model. One of the most important development in these areas is the assumption of stochastic volatility. Wiggins (1987)[32], Hull and White (1987)[19], Scott(1987)[28], Johnson and Shanno (1987)[21], Stein and Stein (1991)[29] use different techniques such as finite-difference, series solution, Monte Carlo simulations, and Feynman-Kac function, and so on to model and test models with stochastic volatility. Heston (1993)[17] introduces the first closed form solution for stochastic volatility models with applications to European-style options.

Stochastic volatility models address some of the issues such as volatility smiles and skewness, as well as large kurtosis in the underlying distribution. Finucane and Tomas (1996)[14] develop a lattice method to price American call options. They use interpolations to find out the option prices on the lattice, so recombining nodes on the tree is not an issue in their model. Furthermore, the model is able to calculate several option prices with different underlying asset prices and volatilities from the same lattice, as long as the time-to-maturity and the strike prices are the same. Hence this method is more efficient than other numerical methods, especially when working with intraday prices or performing scenario analysis.

This paper extends Finucane and Tomas (1996)[14] method in the numerical pricing of American options under stochastic volatility in a non-trivial way involving a three-dimensional lattice. This modified Lattice Based Approach (LBA) is an efficient and accurate numerical method that is not only theoretically sound but also easy to implement. As far as we know, this paper is the first empirical test of a numerical method in pricing American options with stochastic volatility. We are able to show that the new model fits actual prices better, performs better out-of-sample prediction, and also serves as an effective benchmark to signal under- or over-pricing.

Researchers have long been looking for efficient numerical methods to price stochastic volatility models. The direct application of the CRR binomial method would be computationally explosive due to the non-recombining nature of the stochastic volatility model. Moreover, American-style options add additional difficulties as they lack analytical solutions and often require numerical solutions due to the early exercise feature. Cox, Ross and Rubinstein (1979)[13] introduces the standard binomial option-pricing model. Barone-Adesi and Whaley (1987)[4], Geske and Roll (1984)[15] have shown various methods to approximate the early exercise premium under the constant volatility assumption.

There have been many empirical tests of stochastic volatility models. Chesney and Scott (1989)[10] use the stochastic volatility model developed by Scott (1987)[28] to study the prices of dollar/Swiss franc exchange rate. Melino and Turnbull (1990)[23] use a similar model as Scott (1987)[28] to study the Canadian dollar to US dollar options traded in Philadelphia Exchange. Nandi (1996)[25] empirically tests the Heston (1993)[17] model with S&P 500 index options. Bates (1996)[5] tests the stochastic volatility model as part of his stochastic volatility and jumps model using Deutsche mark American options. However, these empirical test either use European options, or use constant volatility model based approximation for early exercise premium plus a stochastic volatility model solution for European options to test American option under stochastic volatility.

Hilliard and Schwartz (1996)[18] develop a transformed recombining binomial tree. Transformations of the variables based on Nelson and Ramaswamy (1990)[26] are performed to make sure that the nodes will recombine. However, the volatility of the process must be in the form of a constant multiple of the instantaneous volatility of the return of the underlying asset. This specification is required in the transformation process to achieve recombination of the tree. The transformation process will increase the magnitude of the underlying asset price and volatility, which could cause the pseudo-probabilities to be negative. In addition, most of the computation time is wasted on those nodes with nearly zero probabilities. Numerical methods for American options with stochastic volatility developed in the past are also less efficient and based on smaller sample size. Our LBA method provides for improvements over these shortcomings.

The rest of this paper is divided into 4 sections. Section 2 derives the stochastic volatility model. Section 3 shows the interpolated lattice-based approach to implement the model. Section 4 reports the empirical tests and the findings of the test results. Section 5 contains the conclusions.

# 2 Model

## 2.1 Underlying Assumptions

The interpolated lattice approach in this paper uses the underlying assumption of Heston (1993)[17]. Define the squared volatility as the instantaneous variance of the return of the underlying asset. The underlying assets and squared volatility are assumed to follow the continuous time stochastic processes:

$$dS = \mu S dt + S\sqrt{V} dz_S \tag{1}$$

$$dV = \kappa(\theta - V)dt + b\sqrt{V} dz_V, \tag{2}$$

where $\mu$ is the expected return of the underlying asset. The squared volatility $V$ is assumed to follow the square root process, with long-run mean $\theta$ and the speed of adjustment $\kappa$. The half-life of the volatility shock is therefor $\ln 2/\kappa$. $b$ is the volatility of the squared volatility process. $\rho$ is the correlation coefficient between the two standard Brownian motions $z_S$ and $z_V$.

Finucane and Tomas (1996)[14] use the assumptions on diffusion processes of Hull and White (1987)[19], in which both the underlying assets and squared volatility are assumed to follow the lognormal process. They then adopt the risk-neutral probability measure and discrete time jumps derived by Boyle (1988)[7], which is an extension of trinomial tree method for one factor model to 5-nomial trees for options on two underlying assets. Boyle (1988) derives the probability measure based on the risk-neutral valuation principle, where the expected returns on both underlying assets are assumed to be the risk-free interest rate. While it is appropriate with traded assets like stocks and futures, it is arguable whether it can be used to price the volatility, which is not traded and unobservable. Many researchers believe that the market price or risk premium of the volatility should be taken into account. As a result, it is not appropriate to directly apply Boyle's probability parameters in the stochastic volatility situation.

While it is difficult to specify the function of the market price of the risk of volatility, a simplified and intuitive way is to assume that it is a linear function of the squared volatility. After adding the market price $\lambda$ to the above mean reverting squared volatility process, it becomes:

$$dV = [\kappa(\theta - V) - \lambda V]dt + \sigma_V \sqrt{V} dz_V. \tag{3}$$

Equation (3) can be reduced to the same initial form as in equation (2) with risk-neutralized parameters:

$$\kappa^* = \kappa + \lambda \qquad \theta^* = \frac{\kappa\theta}{\kappa + \theta}. \tag{4}$$

So when using the square root process, we allow for mean reversion, correlation, and nonzero risk premium in the squared volatility process. However, the parameters $\kappa^*$ and $\theta^*$ estimated from the trading prices cannot be directly interpreted as the speed of adjustment and the long run mean of the squared volatility process.

## 2.2 Risk-Neutral Probabilities and Jumps

To approximate a continuous time process using discrete time jumps and probabilities, the first and second moments of the two processes must be matched. In most options pricing literature, when the underlying asset is assumed to follow a lognormal process, we have the following result:

$$d \ln S = (r - \frac{1}{2}\sigma^2)dt + \sigma dz. \tag{5}$$

We derive in Appendix A the solution for the risk-neutral probabilities and additive jumps (price movements) for our problem setup:

$$\begin{aligned} u_S &= Sr\Delta t + S\sqrt{V\Delta t} \\ d_S &= Sr\Delta t - S\sqrt{V\Delta t} \\ u_V &= \kappa(\theta - V)\Delta t + b\sqrt{V\Delta t} \\ d_V &= \kappa(\theta - V)\Delta t - b\sqrt{V\Delta t} \\ p_{11} &= p22 = 0.25 + 0.25\rho \\ p_{21} &= p12 = 0.25 - 0.25\rho, \end{aligned} \tag{6}$$

where
$u_S$, $u_V$ are upward jumps for $S$ and $V$,
$d_S$, $d_V$ are downward jumps for $S$ and $V$,
$p_{ij}$ is the joint probability of the jumps where i is 1 for up and 2 for down for the underlying asset process, and j represents the squared volatility process.

One nice property of the above solution is that the joint probabilities are independent on the jump sizes, so they need not be calculated at every point on the lattice. The jumps are dependent on $S$, $V$ and other structural parameters. However, the $S$, and $V$ values are associated with points on the lattice and is the same across time, so the jumps need to be calculated also only once. The method is more efficient when compared with other methods attempting to get a recombining tree, in which the jumps and probabilities usually need to be calculated at every node.

In addition, to make sure that the numerical approximation converges to the continuous time counterpart, it is important to ensure that the four joint probabilities are within (0, 1), and together sum to one. This is not a problem when the volatility is assumed to be constant. However, when the volatility follows a stochastic process, normal binomial methods extended from Cox, Ross and Rubinstein's binomial method often lead to probabilities that are negative or greater than one, when probabilities need to be calculated everywhere in the tree. Not only is it costly in computation, a single node with a negative probability will cause the entire tree to break down. The problem is partly inherited from the fact that Cox, Ross, and Rubinstein's tree requires a constraint on the magnitude of volatility. The problem is most severe with methods that use transformations to achieve recombining trees (e.g. the bivariate binomial tree approach in Hilliard and Schwartz (1996)[18]). Our solution avoids this problem.

# 3 Interpolated Lattice Based Approach

The option price at time interval $t$ can be calculated by discounting the expected option prices at time $t+1$. Use $C_t$ to denote the call price at time $t$, $p$ to denote the respective probabilities of future states, and $r_f$ to denote the risk-free rate. Mathematically, the option price is given by the equation:

$$\begin{aligned} C_t(S_t, V_t) = [&p_{11}C_{t+1}(S^+_{t+1}, V^+_{t+1}) + p_{12}C_{t+1}(S^+_{t+1}, V^-_{t+1}) \\ &+ p_{21}C_{t+1}(S^-_{t+1}, V^+_{t+1}) + p_{22}C_{t+1}(S^-_{t+1}, V^-_{t+1})]/(1 + r_f\Delta t). \end{aligned} \tag{7}$$

If the option is American-style, early exercise possibility is checked at every node.

However, a direct application of the above solution using the binomial tree method is computationally explosive. Since the nodes are not recombining, there will be $4^{n+1}$ nodes after $n$ steps. In other words, it is more than 1 trillion nodes after 19 steps. Not only that, most of the computations will be wasted in calculating prices of nearly zero probability, because the upper part of the tree will have asset prices up to millions and the lower part consists of all nearly zeros. However, it is different when we use the interpolated Lattice Based Approach (LBA), because the calculation time only increases linearly with the number of steps chosen, and we can set upper and lower limits of the lattice to improve the efficiency.

Under the lattice based approach, a three dimensional lattice (see Figure 1) is constructed to represent the behavior of the underlying asset prices, and option prices are calculated from the lattice. Use $S^{max}$ and $S^{min}$ to denote the upper and lower limit for the asset price, and $V^{max}$ and $V^{min}$ to denote the limits for the volatility. The continuous range of $S$ and $V$ is then divided into $N_S$ and $N_V$ discrete intervals of length $\Delta S = (S^{max} - S^{min})/N_S$ and $\Delta V = (V^{max} - V^{min})/N_V$. The time-to-maturity is then divided into $N_T + 1$ points of time with intervals of $\Delta t = T/N_t$. At any point of time, $(N_S + 1)(N_V + 1)$ option prices are jointly determined by $N_S+1$ underlying asset prices and $N_V+1$ squared volatility values.

**Figure 1:** Graphical illustration of the Lattice Based Approach

At maturity time $T$, the option price for each $S$ and $V$ is known to be $\max(S - X, 0)$ for calls, and $\max(X - S, 0)$ for puts. So the $(N_S + 1)(N_V + 1)$ option prices at maturity time $T$ are known. For any one of the $(N_S+1)(N_V+1)$ option prices at $t = T-1$, e.g. point $X$ in Figure 1, the option price is determined in equation (7) by option prices at the four points a, b, c, d at time $T$, which often are not on the lattice. We can get the required option prices through interpolation. For example, the option price at point 'a' can be interpolated using the option price of the three points A, B, C, which is on its vertical intercept with the three nearest S prices. The interpolation is applied again using point 1, 2, 3 for A, 4, 5, 6 for B, and 7, 8, 9 for C. Similar interpolation can be applied to obtain option price at point b, c, d as well. For $S$ or $V$ with values falling outside the lattice, extrapolation is applied using the top or bottom three points on the lattice. The approximating polynomial is a weighted-average of the three nearest option prices. The interpolation polynomial is given by the following equation[1]:

$$
\begin{aligned}
C_t(S,V) = & \frac{S - S^{min} - (i+1)\Delta S}{(S^{min} + i\Delta S) - (S^{min} + (i+1)\Delta S)} \cdot \frac{S - S^{min} - (i+2)\Delta S}{(S^{min} + i\Delta S) - (S^{min} + (i+2)\Delta S)} \\
& \cdot C_{t+1}(S^{min} + i\Delta S, V) + \frac{S - S^{min} - i\Delta S}{(S^{min} + (i+1)\Delta S) - (S^{min} + i\Delta S)} \\
& \cdot \frac{S - S^{min} - (i+2)\Delta S}{(S^{min} + (i+1)\Delta S) - (S^{min} + (i+2)\Delta S)} \cdot C_{t+1}(S^{min} + (i+1)\Delta S, V) \\
& + \frac{S - S^{min} - i\Delta S}{(S^{min} + (i+2)\Delta S) - (S^{min} + i\Delta S)} \cdot \frac{S - S^{min} - (i+1)\Delta S}{(S^{min} + (i+2)\Delta S) - (S^{min} + (i+1)\Delta S)} \\
& \cdot C_{t+1}(S^{min} + (i+2)\Delta S, V)
\end{aligned}
\tag{8}
$$

Graphically, the weight assigned for option price at point $A$ is $\frac{\overrightarrow{aB} \cdot \overrightarrow{aC}}{\overrightarrow{AB} \cdot \overrightarrow{AC}}$, where $\overrightarrow{AB}$ is the distance from point $A$ to $B$. The same polynomial is applied three times using option values of point on the lattice (points 1-9) with the nearest V values and S values respectively to get the option price required for equation (8). It is applied one more time to get one of the four required option values used in equation (7). The same interpolation technique is applied at time 0, to enable the method to obtain several option prices with different underlying prices and volatilities at the same time. Since the model uses extrapolation for values that fall outside the lattice, the price of the options are not very sensitive to the boundaries of $S$ and $V$. The discrete time nature of the method also allows it to price American options with discrete dividend payments. Moreover, deltas with respect to the underlying and the squared volatility of the options can be directly calculated from the lattice using two nearest options price of different underlying prices or volatility.

There are potential problems with the interpolation of call prices. Firstly, at the maturity time, the option price is a kink function against underlying asset price, and is thus not differentiable. Secondly, if the slope of option price changes rapidly, which is observed for near the maturity at-the-money options, interpolation may produce larger errors. Interpolation may pose a problem when the call prices are not very smooth with respect to the underlying asset price and the volatility. Lastly, when the jump size is large with respect to the density of the lattice, the extrapolation at the boundary of the lattice may be inaccurate. In addition, the error will multiply itself over the life of the option due to the nature of the method and result in unacceptable option prices. However, we solve these problems to a large extent by checking for appropriate option prices where needed and increasing the number of time intervals to decrease the size of jumps. For example, options which are slightly out-of-the-money often get

[1] The equation used here is the similar to Finucane and Tomas (1996)[14], except that we add the lower bounds to the lattice to improve computational efficiency.

negative values when equation (8) is applied to get the interpolated value. We fix this problem by checking for negative values and setting them to zeros. The extrapolation error will be most significant when the term-to-maturity is long. The options used in this paper are generally short time-to-maturity options, so this is not a problem. However, we find that one possible way to deal with the extrapolation error is to increase the distances among the three points used in the extrapolation. Instead of choosing the top or bottom three point with very small intervals, which often exaggerates the error, we can choose some other points with more appropriate distances between each other.

## 3.1 Validity of LBA

Because interpolation is used, any analytical assessment of the validity of the method will be extremely complicated. A numerical comparison between a known analytical solution is useful to show its numerical convergence when the analytical solution is assumed to be the true price.

Firstly, LBA model is compared with the Black-Scholes Model in valuing a European call option. We shall recognize that LBA is actually a general form of constant volatility models. When the parameters $\kappa$, $\theta$, $b$, and $\rho$ are all set to zero, the LBA is effectively valuing a constant volatility option. In the comparison, $N_S$, $N_V$, and $N_T$ are set to 100, 50, and 200 respectively, $S^{max}$ and $S^{min}$ are set to $1.5 \times X$ and $0.5 \times X$ respectively, and $V^{max}$ and $V^{min}$ are set to $5 \times V$ and 0 respectively[2]. CRR model with 200 steps is also used to price the options. As we can see from Table 1, the two numerical methods' prices are very close to the true price. The average absolute percentage pricing error is around 0.1%.

Secondly, to assess LBA model's ability to price European options with stochastic volatility, it is compared with Heston (1993)'s solution for valuing European options with stochastic volatility. The standard binomial tree method is not used here, because it is not computationally feasible to price options with stochastic volatility. As we can see from Table 2, LBA prices are very close to the true prices calculated using the analytical solution. Except for 3 cases of the out-of-the-money option, the absolute percentage pricing errors are below 0.2%. Since there is no analytical solution available or other known efficient and yet accurate numerical methods, we cannot directly assess the validity of LBA in valuing American options under stochastic volatility. However, it should be consistent with its accuracy in valuing European options.

However, one should note that due to the interpolation, the relationship between the density and the boundary of the lattice and the accuracy of the method is not so straightforward. For example, since there are three dimensions of the lattice, just increasing one dimension will not necessarily increase the accuracy. Our simulated validity test result shows that our choice of the boundaries of the lattice as well as $N_S$, $N_T$, and $N_V$ is reasonably good in terms

[2]Finucane and Tomas (1996) use $2 \times X$ for $S^{max}$ and zero for $S^{min}$, which is inefficient in valuing options with large underlying prices. We follow their boundary set for V, i.e. $5V$ for $V^{max}$ and 0 for $V^{min}$.

**Table 1:** Comparison between BS, CRR, and LBA Models

This table shows and compares European call option prices under constant volatility assumption calculated using different models, namely the Black-Scholes model (BS) the Cox, Ross, and Rubinstein model (CRR) and our Lattice Based Approach (LBA). To illustrate the accuracy of the model, we calculate prices for call options under different moneyness by choosing the underlying stock price from 90 to 110, under different volatilities by choosing $\sigma$ of 0.2, 0.3 and 0.4, and under different time to maturities of 1 month, 3 months and 6 months. Value chosen for common parameters are $X = 100$, $r = 0.05$. The prices are simulated using these parameters. 200 steps are used for both the CRR model and LBA. For LBA, $N_S$ and $N_V$ are set to 100 and 50 respectively. The Pricing Error is measured as percentage of (model price - true price)/true price, taking the BS price as the true price. Mean Error ME, Mean Absolute Error MAE, and Root Mean Squared Error RMSE of the percentages are reported for the entire sample.

| | | T=1/12 | | | T=3/12 | | | T=6/12 | |
|---|---|---|---|---|---|---|---|---|---|
| LBA | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 |
| S=90 | 0.0886 | 0.4896 | 1.1451 | 0.8982 | 2.3063 | 3.9322 | 2.3474 | 4.7143 | 7.2063 |
| S=95 | 0.6552 | 1.5593 | 2.5568 | 2.2753 | 4.1069 | 5.9838 | 4.2559 | 6.9293 | 9.6218 |
| S=100 | 2.5101 | 3.6666 | 4.8082 | 4.6195 | 6.5820 | 8.5547 | 6.8856 | 9.6366 | 12.4094 |
| S=105 | 5.9894 | 6.8851 | 7.9056 | 7.9250 | 9.7130 | 11.6154 | 10.2024 | 12.8006 | 15.5466 |
| S=110 | 10.5206 | 10.9790 | 11.7168 | 11.9877 | 13.4022 | 15.1122 | 14.0729 | 16.3684 | 19.0055 |
| CRR | | | | | | | | | |
| S=90 | 0.0869 | 0.4910 | 1.1476 | 0.8990 | 2.3032 | 3.9358 | 2.3508 | 4.7219 | 7.2091 |
| S=95 | 0.6563 | 1.5537 | 2.5541 | 2.2745 | 4.1114 | 5.9840 | 4.2602 | 6.9342 | 9.6150 |
| S=100 | 2.5092 | 3.6543 | 4.7996 | 4.6100 | 6.5756 | 8.5427 | 6.8817 | 9.6244 | 12.3711 |
| S=105 | 5.9881 | 6.8775 | 7.8985 | 7.9268 | 9.7168 | 11.6187 | 10.2071 | 12.8072 | 15.5105 |
| S=110 | 10.5202 | 10.9800 | 11.7159 | 11.9893 | 13.4079 | 15.1168 | 14.0772 | 16.3663 | 18.9471 |
| Black-Scholes | | | | | | | | | |
| S=90 | 0.0878 | 0.4902 | 1.1451 | 0.8975 | 2.3087 | 3.9316 | 2.3494 | 4.7140 | 7.1993 |
| S=95 | 0.6557 | 1.5534 | 2.5547 | 2.2712 | 4.1053 | 5.9823 | 4.2545 | 6.9282 | 9.6072 |
| S=100 | 2.5121 | 3.6586 | 4.8053 | 4.6150 | 6.5831 | 8.5526 | 6.8887 | 9.6349 | 12.3850 |
| S=105 | 5.9901 | 6.8810 | 7.9036 | 7.9229 | 9.7123 | 11.6130 | 10.2013 | 12.7986 | 15.5057 |
| S=110 | 10.5202 | 10.9799 | 11.7163 | 11.9883 | 13.4025 | 15.1097 | 14.0754 | 16.3655 | 18.9359 |
| Error LBA | | | | | | | | | |
| S=90 | 0.90% | -0.12% | -0.00% | 0.07% | -0.10% | 0.01% | -0.08% | 0.01% | 0.10% |
| S=95 | -0.08% | 0.38% | 0.08% | 0.18% | 0.04% | 0.03% | 0.03% | 0.02% | 0.15% |
| S=100 | -0.08% | 0.22% | 0.06% | 0.10% | -0.02% | 0.02% | -0.05% | 0.02% | 0.20% |
| S=105 | -0.01% | 0.06% | 0.03% | 0.03% | 0.01% | 0.02% | 0.01% | 0.02% | 0.26% |
| S=110 | 0.00% | -0.01% | 0.00% | -0.01% | -0.00% | 0.02% | -0.02% | 0.02% | 0.37% |
| Mean | 0.06% | MAE | 0.09% | RMSE | 0.18% | | | | |
| Error CRR | | | | | | | | | |
| S=90 | -0.98% | 0.17% | 0.22% | 0.17% | -0.24% | 0.10% | 0.06% | 0.17% | 0.14% |
| S=95 | 0.09% | 0.01% | -0.02% | 0.14% | 0.15% | 0.03% | 0.13% | 0.09% | 0.08% |
| S=100 | -0.11% | -0.12% | -0.12% | -0.11% | -0.11% | -0.12% | -0.10% | -0.11% | -0.11% |
| S=105 | -0.03% | -0.05% | -0.06% | 0.05% | 0.05% | 0.05% | 0.06% | 0.07% | 0.03% |
| S=110 | -0.00% | 0.00% | -0.00% | 0.01% | 0.04% | 0.05% | 0.01% | 0.01% | 0.06% |
| Mean | 0.00% | MAE | 0.10% | RMSE | 0.18% | | | | |

**Table 2:** Comparison between Heston and LBA Models
This table shows and compares European call option prices under stochastic volatility using the Heston model and the Lattice Based Approach (LBA). Standard models like BS and CRR cannot handle stochastic volatility. Values chosen for common parameters are $X = 100$, $r = 0.05$, $t = (1/12, 3/12, 6/12)$, $\sqrt{V} = (0.2, 0.3, 0.4)$, $\kappa = 3$, $\theta = 0.04$, $b = 0.1$, $\rho = -0.1$. For LBA, $N_S = 100$, $N_T = 200$, and $N_V = 50$. The prices are simulated using these parameters. The Pricing Error is measured as percentage of (model price - true price)/true price, taking the Heston price as the true price. Mean Error ME, Mean Absolute Error MAE, and Root Mean Square Error RMSE of the percentages are reported for the entire sample.

| | | T=1/12 | | | T=3/12 | | | T=6/12 | |
|---|---|---|---|---|---|---|---|---|---|
| LBA | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 | $\sigma$=0.2 | $\sigma$=0.3 | $\sigma$=0.4 |
| S=90 | 0.0865 | 0.4331 | 1.0123 | 0.8859 | 1.9028 | 3.1373 | 2.3263 | 3.6462 | 5.1797 |
| S=95 | 0.6504 | 1.4623 | 2.3730 | 2.2631 | 3.6178 | 5.0855 | 4.2374 | 5.7545 | 7.4444 |
| S=100 | 2.5088 | 3.5552 | 4.6061 | 4.6151 | 6.0719 | 7.6212 | 6.8810 | 8.4396 | 10.1737 |
| S=105 | 5.9938 | 6.7944 | 7.7231 | 7.9310 | 9.2412 | 10.7125 | 10.2071 | 11.6580 | 13.3332 |
| S=110 | 10.5244 | 10.9245 | 11.5778 | 11.9998 | 13.0099 | 14.2906 | 14.0896 | 15.3362 | 16.8742 |
| Heston | | | | | | | | | |
| S=90 | 0.0857 | 0.4335 | 1.0115 | 0.8852 | 1.9023 | 3.1355 | 2.3272 | 3.6447 | 5.1748 |
| S=95 | 0.6509 | 1.4551 | 2.3696 | 2.2588 | 3.6147 | 5.0826 | 4.2367 | 5.7520 | 7.4382 |
| S=100 | 2.5107 | 3.5455 | 4.6017 | 4.6105 | 6.0703 | 7.6177 | 6.8817 | 8.4366 | 10.1664 |
| S=105 | 5.9944 | 6.7897 | 7.7198 | 7.9290 | 9.2385 | 10.7087 | 10.2068 | 11.6550 | 13.3245 |
| S=110 | 10.5241 | 10.9253 | 11.5766 | 12.0006 | 13.0087 | 14.2873 | 14.0910 | 15.3337 | 16.8632 |
| Percentage Error | | | | | | | | | |
| S=90 | 0.96% | -0.09% | 0.08% | 0.08% | 0.02% | 0.06% | -0.04% | 0.04% | 0.09% |
| S=95 | -0.07% | 0.49% | 0.15% | 0.19% | 0.09% | 0.06% | 0.02% | 0.04% | 0.08% |
| S=100 | -0.08% | 0.27% | 0.10% | 0.10% | 0.03% | 0.05% | -0.01% | 0.04% | 0.07% |
| S=105 | -0.01% | 0.07% | 0.04% | 0.03% | 0.03% | 0.03% | 0.00% | 0.03% | 0.06% |
| S=110 | 0.00% | -0.01% | 0.01% | -0.01% | 0.01% | 0.02% | -0.01% | 0.02% | 0.07% |
| Mean | 0.07% | MAE | 0.09% | RMSE | 0.18% | | | | |

of both accuracy and computational speed[3].

Lastly, we show graphically the effect of stochastic volatility on (American) options in Figure 2. The upper-left graph shows us the surface of European call option prices over time and across different moneyness. The European call options are calculated using the Black-Scholes formula using $X = 100$, $r = 0.05$, $\sigma = 0.2$ and a dividend yield of 0.05. $S$ ranges from 90 to 110, and time-to-maturity from 0.5 to 0. Using a dividend yield that is equal to the risk-free interest rate, it is the same as valuing a futures option. As we can see, at the maturity ($t = T$), the option value is actually a kink function of $S$. As the time-to-maturity increases, the kinky line smoothes out and the time value increases. If we look carefully enough, we can also see that the time value of at-the-money option ($S = X = 100$) is the largest compared to in-the-money or out-of-the money options.

When taking stochastic volatility into consideration, the surface of the option value is basically the same as the constant volatility value, since the differences are small with respect to the option values, so we did not show a separate graph. However, when we just look at the differences, there are some interesting patterns. The upper-right graph shows us the differences of the European call options value with stochastic volatility minus the option values with constant volatility. For the additional parameters of LBA, we use $k = 1$, $\theta = 0.04$, $b = 0.3$, $\rho = -0.3$, $N_T = N_V = 50$, $N_S = 100$. We can see that the constant

[3]The algorithms are implemented partly in Matlab and partly in C. LBA model costs around 1 second on a Pentium II 333Mhz PC for one 3D lattice simulation.

**Figure 2:** Effects of stochastic volatility in pricing (American) options

volatility model systematically underprices in-the-money calls and overprices out of the money calls with respect to stochastic volatility models. The degree of mispricing increases proportionately with the time-to-maturity and the absolute moneyness. The findings are just the opposite when the correlation is positive, as shown in the lower-left graph. The constant volatility model overprices in-the-money calls and underprices out-of-the-money options.

The lower-right graph of Figure 2 shows us the early exercise premium of the American option over its European counterpart. As we can see, the premium decreases as the option approaches the maturity and is largest for long time-to-maturity and in-the-money options.

# 4 Empirical test

We empirically test the LBA model. As a comparison, the CRR model is used to perform an empirical test of the constant volatility model.

## 4.1 Empirical Data

The empirical test of the model is performed with intraday trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from January 4, 1995 to March 16, 1995. We choose the data based on two considerations. Firstly, the options are actively traded American options. Secondly, S&P 500 index and S&P500 futures options are widely used in existing literature. Both the underlying futures and the options on futures expired on March 16, 1995. There are very few trades before the sample period.

Option prices are matched with nearest corresponding futures price preceding the option transaction. Both the futures and the options prices are quoted in index points. The futures prices in the sample period range from 460.35 to 492.1, and the most actively traded options over the life of the futures are at-the-money options, which have strike prices ranging from 475 to 495. There are 3742 call option prices in the sample. Several common filtering rules are applied to the raw sample before the data are used in the study.

Firstly, call prices that are less than six days to maturity are taken out of the sample to avoid possible liquidity related bias. In fact, we observe many cases in the last few days when the call prices move in the opposite direction of the underlying asset price movements. This can only be explained as relatively large intraday fluctuations of the volatility, which may potentially distort the parameter estimation. This is one of the filtering rules used by Bakshi et al (1997)[2].

Secondly, call prices that are less than 0.5 are not used to mitigate the impact of price discreteness (the tick size for S&P 500 futures option is 0.05). This is because most option pricing models assume continuous price movements, while in real world the prices move in ticks. Nandi (1996)[25] use this rule with his S&P index options data.

Lastly, the options with time lag of more than 30 seconds with the corresponding futures are eliminated to ensure the concurrence between the option and its underlying asset. A similar rule is applied in Chang, Chang and Lim (1998)[9]. Due to the more frequent trading of the futures, this rule does not exclude many options from the sample. The filtered sample consists of 2,957 call prices. A more detailed breakdown of the raw data and filtered data statistics according to moneyness and time-to-maturity is shown in Table 3.

The risk-free rate is calculated from quotes of US Treasury Bill prices best matching the options maturity. The US Treasury Bill used in this study matured on March 23, 1995. The quotes are obtained through Reuters Fixed Income 3000. The average of bid and ask discount quotes is used to calculate the yield.[4]

## 4.2 Parameter Estimation

To implement option pricing models, the spot volatility and other unobservable structural parameters need to be backed out implicitly using the observed traded option prices on each day. There is some internal inconsistency between the models and this daily estimation, since the models here assume that the structural parameters are constant. However, as daily estimations greatly increase the fit, we follow many existing empirical studies to use this method.[5]

[4]There are four missing data in the 46 sample days, where the averages of the two nearest yields are used.

[5]see Bakshi et al (1997)[2] and Chang, Chang and Lim (1998)[9]

**Table 3:** Sample statistics

The sample data are intraday trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from Jan 4, 1995 to Mar 16, 1995. They are divided into 3 time-to-maturity and 6 moneyness groups, so we have $3+6=9$ subgroups, $3 \times 6 = 18$ subsubgroups, and one grand total, altogether 28 groups. Number of observations, mean and standard deviation are given for each group.

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| $< 0.98$ | | | | | | | | |
| Number | 401 | 388 | 67 | 65 | 61 | 6 | 529 | 459 |
| Mean | 1.804 | 1.836 | 1.170 | 1.174 | 0.385 | 0.608 | 1.560 | 1.726 |
| Standard Deviation | 0.767 | 0.752 | 0.197 | 0.198 | 0.101 | 0.038 | 0.822 | 0.743 |
| $0.98 - 0.99$ | | | | | | | | |
| Number | 241 | 239 | 295 | 282 | 250 | 212 | 786 | 733 |
| Mean | 3.031 | 3.026 | 1.909 | 1.903 | 0.945 | 0.997 | 1.947 | 2.007 |
| Standard Deviation | 0.762 | 0.749 | 0.516 | 0.515 | 0.293 | 0.271 | 0.993 | 0.972 |
| $0.99 - 1.00$ | | | | | | | | |
| Number | 194 | 189 | 391 | 384 | 610 | 376 | 1,195 | 949 |
| Mean | 4.782 | 4.788 | 3.539 | 3.537 | 1.813 | 2.322 | 2.860 | 3.305 |
| Standard Deviation | 0.654 | 0.657 | 0.789 | 0.786 | 0.906 | 0.706 | 1.415 | 1.173 |
| $1.00 - 1.01$ | | | | | | | | |
| Number | 71 | 70 | 257 | 250 | 474 | 215 | 802 | 535 |
| Mean | 7.005 | 7.012 | 5.958 | 5.965 | 3.646 | 4.679 | 4.684 | 5.585 |
| Standard Deviation | 0.754 | 0.757 | 0.973 | 0.977 | 1.465 | 1.059 | 1.804 | 1.278 |
| $1.01 - 1.02$ | | | | | | | | |
| Number | 42 | 41 | 92 | 89 | 153 | 55 | 287 | 185 |
| Mean | 10.255 | 10.271 | 9.150 | 9.140 | 7.323 | 7.963 | 8.338 | 9.041 |
| Standard Deviation | 0.864 | 0.869 | 1.121 | 1.124 | 1.459 | 1.422 | 1.715 | 1.432 |
| $\geq 1.02$ | | | | | | | | |
| Number | - | - | 74 | 74 | 69 | 22 | 143 | 96 |
| Mean | NaN | NaN | 13.049 | 13.049 | 12.812 | 11.691 | 12.935 | 12.738 |
| Standard Deviation | NaN | NaN | 1.130 | 1.130 | 2.688 | 0.907 | 2.032 | 1.221 |
| All Moneyness | | | | | | | | |
| Number | 949 | 927 | 1,176 | 1,144 | 1,617 | 886 | 3,742 | 2,957 |
| Mean | 3.488 | 3.509 | 4.561 | 4.582 | 3.153 | 3.148 | 3.680 | 3.816 |
| Standard Deviation | 2.260 | 2.250 | 3.168 | 3.184 | 2.980 | 2.431 | 2.942 | 2.769 |

For the LBA model, five parameters of the process $V$, $\kappa$, $\theta$, $b$, and $\rho$ need to be implied from the traded option prices. For the CRR model, only the volatility needs to be implied.[6]

There are several possible objective function candidates to be minimized in order to imply out the volatility and other necessary parameters. For example, the parameters can be estimated by minimizing the sum of squared percentage pricing error. This method will put more weight on matching the cheaper options since the same dollar pricing error will lead to larger percentage pricing errors on cheaper options. Alternatively, we can minimize the sum of squared index point pricing error. However, this will lead to more weight on matching the more expensive options. We follow the convention in most research to minimize the sum of squared index point error.

The nonlinear optimization result achieved using the Sequential Quadratic Programming method does not ensure a global minimum of the objective function, and to some extent depends on the initial guesses. Since our objective surface is very irregular and we are estimating five variables at the same time, the estimated parameters are very sensitive to the initial values. To ensure stability of the estimates, we add a constraint on the long run mean and the volatility:$\theta - \sigma^2 < 0.05$. This is not a restrictive constraint since the estimated

[6]The algorithms of the LBA are implemented in C Programming Language for computational efficiency. The optimization procedure uses Sequential Quadratic Programming methods in Matlab.

$\sigma$ is about 0.09. After we imposed the constraint, the sample implied parameters appeared to be in order. Due to the computational time constraint, we perform a two-step estimation. Firstly, we estimated the parameters using the results of previous studies with similar data as the initial values. After that, we re-estimated the parameters using the average of the previous estimates as the initial values.

The same method is used to get the daily-implied volatility of the standard binomial tree model for comparison purpose.

The estimates of parameters of the LBA and CRR model are reported in Table 4.

**Table 4:** Parameter estimation results

| | CRR | LBA | | | | |
|---|---|---|---|---|---|---|
| | $\sigma$ | $\sqrt{V}$ | $b$ | $\kappa$ | $\theta$ | $\rho$ |
| Mean | 0.10 | 0.09 | 0.61 | 3.04 | 0.03 | -0.52 |
| Std Deviation | 0.01 | 0.01 | 0.19 | 0.90 | 0.01 | 0.13 |

As we mentioned before, the risk-neutralized theta cannot be directly interpreted as the long run mean of volatility. However, using equation (4), we can infer that the true long-run mean will be larger than our estimates for any positively defined, unobservable market price of the risk of volatility . We find that the long run mean of the squared volatility, theta, is always larger than the squared volatility. This implies that the volatility has a tendency to increase during the life of the futures. This is also reflected by the lower mean of the implied volatilities of LBA relative to the CRR, because the implied volatility for LBA is just the initial volatility, and that of the CRR is the average volatility over the life of the option. We can see from Figure 3 the increasing trend in the implied volatility time series of LBA. To verify this point, we run a simple OLS regression of the implied volatility over the days to maturity. The coefficient is negative (indicating an increasing trend of the volatility), and statistically different from zero at 1% significance level for both the LBA and CRR.
The half-life of the volatility shock as measured in days is $\frac{365 \cdot \ln 2}{\kappa}$, or 83.1 days.

## 4.3 Tests of Model Performance

The model's performance is examined using four tests: in-sample test, out-of-sample test, dynamic intraday hedging performance test and profitability test. We divide the moneyness into six sub categories and the time-to-maturity into three. Together with the subtotals and grand totals, there are 28 categories. For the first three tests, we reported the Mean error, Mean of Absolute Error (MAE) and Root Mean Square Error (RMSE) statistics of both in terms of index points as well as percentage to assess different aspects of the error. For the profitability test, the mean profit/loss for each category is reported. The index

**Figure 3:** Implied volatility estimates

point error is defined as the difference between actual price and model price, and the percentage error is the index point error divided by the model price. Some studies base the percentage error on the actual price. Mathematically,

$$\epsilon_i \equiv C_a - C_m \tag{9}$$

$$\epsilon_{\%} \equiv \frac{C_a - C_m}{C_m} \times 100\% \tag{10}$$

where $C_a$ is the actual option price, $C_m$ is the model option price, $\epsilon_i$ is the index point error, and $\epsilon_{\%}$ is the percentage error.

### 4.3.1 In-Sample Test

The in-sample test is to assess the goodness-of-fit of the model price compared to actual prices, using the parameter estimated employing all the data in the same day. We expect that the stochastic volatility model will fit the prices better simply because it uses more parameters and therefore allows for more degree of freedom. Moneyness and Time-To-Maturity biases are examined to assess the extent of model misspecification.

The in-sample test results are shown in Table 5. The LBA model shows better goodness-of-fit with no evidence of bias related to moneyness or maturity. In contrast, the CRR model displays evidence of moneyness bias.

Bias. The overall in-sample performances show no significant bias for both models. The mean errors in index point terms and percentage terms of all call options in the sample for both models are very small. There is no significant maturity bias for both models. This is partly due to the relatively overall short time-to-maturity of the options on futures. However, we notice that the CRR model displays systematic bias in terms of moneyness. In every time-to-maturity category, the CRR model overprices (shown by a negative number) out

**Table 5:** In-Sample Test Result

The in-sample test result is split into two panels. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data in the same day.

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| **Panel A: Index Points Error** | | | | | | | | |
| $< 0.98$ | | | | | | | | |
| Mean | -0.142 | 0.003 | -0.176 | 0.009 | -0.026 | 0.062 | -0.146 | 0.004 |
| MAE | 0.185 | 0.064 | 0.176 | 0.050 | 0.049 | 0.064 | 0.182 | 0.062 |
| RMSE | 0.210 | 0.086 | 0.194 | 0.063 | 0.055 | 0.074 | 0.206 | 0.083 |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | -0.081 | -0.011 | -0.179 | -0.005 | -0.104 | 0.005 | -0.125 | -0.004 |
| MAE | 0.187 | 0.073 | 0.179 | 0.062 | 0.113 | 0.052 | 0.163 | 0.063 |
| RMSE | 0.214 | 0.095 | 0.196 | 0.076 | 0.132 | 0.064 | 0.186 | 0.079 |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | 0.091 | 0.004 | -0.088 | 0.007 | -0.062 | -0.001 | -0.042 | 0.003 |
| MAE | 0.170 | 0.075 | 0.123 | 0.070 | 0.106 | 0.083 | 0.125 | 0.076 |
| RMSE | 0.208 | 0.096 | 0.143 | 0.089 | 0.131 | 0.106 | 0.154 | 0.097 |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 0.245 | 0.008 | 0.128 | -0.008 | 0.117 | -0.008 | 0.139 | -0.006 |
| MAE | 0.248 | 0.087 | 0.146 | 0.095 | 0.189 | 0.118 | 0.177 | 0.103 |
| RMSE | 0.281 | 0.117 | 0.191 | 0.120 | 0.232 | 0.146 | 0.222 | 0.131 |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | 0.550 | 0.003 | 0.371 | -0.022 | 0.299 | 0.021 | 0.389 | -0.004 |
| MAE | 0.550 | 0.099 | 0.374 | 0.106 | 0.324 | 0.133 | 0.398 | 0.112 |
| RMSE | 0.568 | 0.146 | 0.414 | 0.134 | 0.394 | 0.164 | 0.447 | 0.146 |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 0.544 | 0.046 | 0.383 | 0.024 | 0.507 | 0.041 |
| MAE | NaN | NaN | 0.544 | 0.130 | 0.389 | 0.088 | 0.509 | 0.120 |
| RMSE | NaN | NaN | 0.570 | 0.169 | 0.440 | 0.119 | 0.543 | 0.159 |
| All Moneyness | | | | | | | | |
| Mean | -0.019 | 0.000 | 0.009 | 0.001 | 0.005 | 0.001 | -0.001 | 0.001 |
| MAE | 0.203 | 0.072 | 0.191 | 0.079 | 0.148 | 0.087 | 0.182 | 0.079 |
| RMSE | 0.243 | 0.096 | 0.247 | 0.103 | 0.197 | 0.114 | 0.232 | 0.105 |
| **Panel B: Percentage Error** | | | | | | | | |
| $< 0.98$ | | | | | | | | |
| Mean | -9.31% | 0.10% | -12.62% | 1.21% | -3.31% | 12.21% | -9.70% | 0.42% |
| MAE | 10.77% | 3.84% | 12.62% | 4.47% | 7.65% | 12.59% | 10.99% | 4.04% |
| RMSE | 12.90% | 5.26% | 13.53% | 5.69% | 8.37% | 14.97% | 12.94% | 5.56% |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | -3.44% | -0.37% | -9.01% | -0.45% | -9.06% | 0.82% | -7.21% | -0.06% |
| MAE | 6.22% | 2.45% | 9.02% | 3.32% | 10.32% | 5.59% | 8.48% | 3.69% |
| RMSE | 7.18% | 3.15% | 10.01% | 4.09% | 11.90% | 7.04% | 9.81% | 4.90% |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | 1.75% | 0.16% | -2.80% | 0.17% | -2.69% | -0.11% | -1.85% | 0.06% |
| MAE | 3.58% | 1.61% | 3.61% | 2.01% | 4.97% | 4.03% | 4.14% | 2.73% |
| RMSE | 4.35% | 2.14% | 4.31% | 2.52% | 6.45% | 5.44% | 5.27% | 3.90% |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 3.56% | 0.15% | 2.09% | -0.02% | 2.12% | -0.20% | 2.30% | -0.07% |
| MAE | 3.61% | 1.29% | 2.42% | 1.61% | 4.01% | 2.66% | 3.22% | 1.99% |
| RMSE | 4.09% | 1.75% | 3.07% | 2.06% | 4.78% | 3.36% | 3.98% | 2.63% |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | 5.68% | 0.05% | 4.14% | -0.26% | 3.70% | 0.21% | 4.35% | -0.05% |
| MAE | 5.68% | 0.98% | 4.18% | 1.16% | 4.05% | 1.71% | 4.48% | 1.28% |
| RMSE | 5.88% | 1.47% | 4.53% | 1.47% | 4.82% | 2.12% | 4.94% | 1.69% |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 4.37% | 0.36% | 3.41% | 0.21% | 4.15% | 0.32% |
| MAE | NaN | NaN | 4.38% | 0.99% | 3.46% | 0.75% | 4.17% | 0.93% |
| RMSE | NaN | NaN | 4.58% | 1.27% | 3.90% | 1.01% | 4.44% | 1.22% |
| All Moneyness | | | | | | | | |
| Mean | -3.91% | -0.01% | -2.81% | 0.01% | -2.50% | 0.20% | -3.06% | 0.06% |
| MAE | 7.36% | 2.71% | 5.29% | 2.25% | 5.94% | 3.90% | 6.14% | 2.89% |
| RMSE | 9.46% | 3.93% | 6.81% | 3.05% | 7.71% | 5.39% | 7.99% | 4.14% |

of money calls and underprices in-the-money calls, and the degree of mispricing is proportional to the moneyness. The bias is evident according to both the mean errors for index points and percentage. This finding is consistent with Macbeth and Merville (1979)[22].

As we expected, the moneyness bias disappears when we take into account the stochastic volatility. There is no obvious pattern of moneyness related bias for the LBA model. In addition, the LBA produces a much smaller bias. The CRR model's degree of bias is generally 10 times more than that of the LBA model.

We also find that the relative magnitude of the mean absolute index points pricing error with respect to its mean index points pricing error is larger in each sub-category for the LBA than the CRR model. For example, in the first category in Table 5 Panel A, where $T >= 35$ and $S/X < 0.98$, CRR produces an MAE of 0.18 and a Mean of -0.14, while LBA has 0.06 and 0.003 respectively. If we take MAE/Mean, we obtain 1.3 for CRR and 20 for LBA. This can also be interpreted as the evidence of CRR model's bias. When most of the errors are of the same sign within each moneyness category, the mean index points error will be close to the mean absolute index points error, and vice versa. The LBA model's pricing error is more evenly distributed around the true price, so the mean index points error is much smaller than the mean absolute index points error.

Goodness-of-fit. The overall mean absolute index points pricing error of CRR is 0.18, while it is 0.08 for the LBA. In percentage terms, the mean absolute error is 2.9% for LBA and 6.1% for the CRR model. The findings in the RMSE measure are consistent with this. Furthermore, the LBA dominates the CRR in almost all the categories.

We report the MAE as well as RMSE, both measuring the model's goodness-of-fit. MAE gives us the sense of what is the average error to expect. On the other hand, RMSE punishes larger errors because larger errors get more weights when we square them, so it is more appropriate for use with risk-averse investors. In both measures, the LBA shows better goodness-of-fit as we expected. The findings in terms of the index points are consistent with the percentage error.

Implications. The smaller bias and the better goodness-of-fit in the in-sample test for LBA over CRR show that there are some evidence of stochastic volatility in the S&P 500 futures options. The stochastic volatility model is a better specification than the constant volatility model. However, we are still not sure whether the better performance of LBA is only an illusion due to more degrees of freedom from more parameters. The question is answered by the result from the out-of-sample test.

#### 4.3.2 Out-of-Sample Test

We test the out-of-sample goodness-of-fit using lagged parameter estimation. In this test, we use the previous trading day's implied volatility and parameter estimates to price the current day's options. This is an important test of model misspecification because if there is no stochastic term in the volatility, the over-fitting in the in-sample test of the stochastic volatility model will be penalized. Since the previous day's data is used to estimate the parameters, this test can also be interpreted as the prediction test of the models and test of parameter stability.

Table 6 shows the out-of-sample test results. Unlike the in-sample test where the LBA dominates in all categories, the results is somewhat mixed in out-sample test. The LBA still displays much smaller bias and better fit. However, in some of the short time-to-maturity and out-of-the-money categories, the results are mixed in terms of MAE and RMSE.

Bias. The CRR model still shows moneyness bias, i.e. increasing degree of overpricing (under pricing) with increasing degree of in-the-money (out of money). The level of mispricing is around the same level as the in-sample test. There is some inconsistency of the mean error measure in index points and percentage. For example, the CRR model underprices the overall call options by 0.01 index points, while it is overpricing the call options by 2.3%. This is not surprising, because the index points measure puts more weight on in-the-money options, which are usually more expensive, and the percentage measure favors out of the money options. From the difference in the two measures, we can also find out that the CRR underprices in-the-money options and overprices out-of-the-money ones.

The LBA generally produces very small bias regardless of different categories. It outperforms the CRR model in every category. However, the mean error in percentage terms increased from 0.06% in the in-sample test to 0.87% in this out-of-sample test.

Goodness-of-fit. The goodness-of-fit for CRR only decreases marginally from the in-sample test. However, the LBA overall error is more than doubled according all measures. This is partly due to the fact that the parameter estimates are less stable for the LBA model.

The LBA out-of-sample fit or prediction performance outperforms the CRR in every major moneyness category, as well as in the overall category. However, a detailed look at each sub-category reveals that the CRR model performs slightly better with out of the money and short time-to-maturity options. The CRR performs better than the LBA for calls with moneyness less than 0.98 and time-to-maturity shorter than 15 days. However, we should notice that there are only 6 calls in this category. In terms of RMSE of percentage error, CRR performs slightly better with calls of moneyness 0.98–0.99 and 0.99–1.00 and time-to-maturity less than 15 days. However, the converse is true in terms of MAE in percentage terms. In addition, the measure in terms of index points shows that the LBA actually dominates the CRR for options with $T \geq 15$ and $S/X$ between 0.98–0.99. This inconsistency may be due to the fact that the percentage error uses the model price as 100% . We can see from the mean percentage error in

**Table 6:** Out-of-Sample Test Result

The out-of-sample test result is split into two panels. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data of the pervious trading day.

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| **Panel A: Index Points Error** | | | | | | | | |
| $< 0.98$ | | | | | | | | |
| Mean | -0.168 | -0.001 | -0.111 | 0.084 | 0.016 | 0.057 | -0.158 | 0.013 |
| MAE | 0.217 | 0.133 | 0.148 | 0.107 | 0.040 | 0.069 | 0.205 | 0.128 |
| RMSE | 0.249 | 0.167 | 0.168 | 0.128 | 0.068 | 0.101 | 0.237 | 0.161 |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | -0.085 | -0.027 | -0.156 | 0.024 | -0.067 | 0.038 | -0.107 | 0.011 |
| MAE | 0.191 | 0.152 | 0.182 | 0.110 | 0.127 | 0.098 | 0.169 | 0.120 |
| RMSE | 0.229 | 0.186 | 0.212 | 0.146 | 0.148 | 0.126 | 0.202 | 0.155 |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | 0.098 | -0.037 | -0.090 | 0.001 | -0.046 | 0.015 | -0.035 | -0.001 |
| MAE | 0.175 | 0.156 | 0.177 | 0.147 | 0.174 | 0.166 | 0.175 | 0.156 |
| RMSE | 0.230 | 0.205 | 0.216 | 0.190 | 0.202 | 0.206 | 0.214 | 0.199 |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 0.276 | -0.074 | 0.163 | 0.016 | 0.189 | 0.052 | 0.188 | 0.019 |
| MAE | 0.299 | 0.231 | 0.231 | 0.195 | 0.282 | 0.218 | 0.261 | 0.209 |
| RMSE | 0.328 | 0.301 | 0.284 | 0.240 | 0.354 | 0.278 | 0.320 | 0.264 |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | 0.597 | -0.081 | 0.372 | -0.052 | 0.306 | -0.067 | 0.402 | -0.063 |
| MAE | 0.597 | 0.228 | 0.378 | 0.170 | 0.373 | 0.259 | 0.425 | 0.209 |
| RMSE | 0.631 | 0.304 | 0.438 | 0.211 | 0.453 | 0.323 | 0.491 | 0.270 |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 0.536 | -0.001 | 0.374 | -0.041 | 0.499 | -0.010 |
| MAE | NaN | NaN | 0.536 | 0.153 | 0.385 | 0.132 | 0.501 | 0.148 |
| RMSE | NaN | NaN | 0.565 | 0.186 | 0.439 | 0.171 | 0.539 | 0.182 |
| All Moneyness | | | | | | | | |
| Mean | -0.023 | -0.025 | 0.024 | 0.011 | 0.039 | 0.023 | 0.014 | 0.004 |
| MAE | 0.225 | 0.155 | 0.227 | 0.148 | 0.205 | 0.167 | 0.220 | 0.156 |
| RMSE | 0.276 | 0.201 | 0.286 | 0.191 | 0.266 | 0.219 | 0.277 | 0.203 |
| **Panel B: Percentage Error** | | | | | | | | |
| $< 0.98$ | | | | | | | | |
| Mean | -10.26% | 0.40% | -6.75% | 8.74% | 4.27% | 13.96% | -9.55% | 1.81% |
| MAE | 12.31% | 8.29% | 11.69% | 10.60% | 8.00% | 15.93% | 12.16% | 8.74% |
| RMSE | 14.76% | 10.83% | 13.11% | 12.98% | 14.60% | 26.02% | 14.52% | 11.51% |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | -3.39% | -0.68% | -7.75% | 1.24% | -4.66% | 5.90% | -5.44% | 1.96% |
| MAE | 6.28% | 4.99% | 9.35% | 6.08% | 12.13% | 11.89% | 9.16% | 7.40% |
| RMSE | 7.51% | 6.03% | 11.05% | 8.21% | 14.01% | 16.15% | 11.05% | 10.64% |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | 1.97% | -0.62% | -2.88% | 0.02% | -2.22% | 0.81% | -1.65% | 0.20% |
| MAE | 3.77% | 3.35% | 5.19% | 4.35% | 8.16% | 8.10% | 6.08% | 5.64% |
| RMSE | 5.05% | 4.43% | 6.49% | 5.71% | 9.87% | 10.64% | 7.79% | 7.87% |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 4.10% | -0.92% | 2.82% | 0.55% | 3.71% | 0.93% | 3.35% | 0.51% |
| MAE | 4.47% | 3.34% | 4.08% | 3.52% | 6.24% | 4.96% | 5.00% | 4.08% |
| RMSE | 4.95% | 4.36% | 5.06% | 4.61% | 7.73% | 6.40% | 6.26% | 5.38% |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | 6.15% | -0.83% | 4.16% | -0.61% | 3.81% | -1.07% | 4.50% | -0.79% |
| MAE | 6.15% | 2.29% | 4.23% | 1.89% | 4.78% | 3.37% | 4.82% | 2.42% |
| RMSE | 6.51% | 3.10% | 4.83% | 2.37% | 5.73% | 4.27% | 5.51% | 3.20% |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 4.31% | 0.00% | 3.34% | -0.36% | 4.09% | -0.08% |
| MAE | NaN | NaN | 4.31% | 1.19% | 3.43% | 1.12% | 4.11% | 1.17% |
| RMSE | NaN | NaN | 4.55% | 1.44% | 3.92% | 1.45% | 4.41% | 1.44% |
| All Moneyness | | | | | | | | |
| Mean | -4.08% | -0.25% | -2.04% | 0.88% | -0.81% | 2.00% | -2.30% | 0.87% |
| MAE | 8.08% | 5.75% | 6.21% | 4.56% | 8.32% | 7.83% | 7.42% | 5.91% |
| RMSE | 10.62% | 7.96% | 7.92% | 6.51% | 10.33% | 11.23% | 9.56% | 8.61% |

these two categories that the CRR produces negative values of -4.7% and -2.2%, i.e. overprices the options, while LBA produces positive values, or overprices the options. Thus the percentage measure is more favorable to the CRR model. The RMSE punishes extreme values more, so it shows that the CRR actually performs better.

Implications. The better out-of-sample test result of LBA verifies again that the stochastic volatility model is a better specification. However, the fact that the out-of-sample errors of LBA doubled while the CRR only increase marginally indicate that the LBA model's parameter is less stable than CRR. Nevertheless, we expect that better parameter estimation procedures may improve the parameter stability of LBA.

#### 4.3.3 Dynamic Intraday Hedging Test

It is often the hedging effectiveness that makes option-pricing models useful to investors. To assess the models' hedging effectiveness, we follow the dynamic intraday hedging performance method of Chang, Chang and Lim (1998)[9]. In their paper, they assume that the hedge portfolio is continuously rebalanced throughout the day. The hedging error is defined as the per tick change in actual price versus change in the model price. This hedge test is different from the traditional daily-rebalanced delta-neutral hedge in the sense that it can capture the intraday dynamics of the prices. The hedging results are shown in Table 7. Both models show reasonably good hedging result without any significant bias. In terms of hedging effectiveness, the two models produce very close errors in terms of MAE as well as RMSE in most categories. However, CRR slightly outperforms LBA in some categories.

Bias. The hedging bias is measured by the mean hedging errors. As we can see from Table 7, the bias is below 0.01 in terms of index points and below 1% in terms of percentage for most categories. The largest error occurs for options with less than 15 days to maturity and with moneyness less than 0.98. The large error is most likely due to the fact that there are only 6 options in this category, hence the category has small weight in the parameter estimation.

Hedging Effectiveness. CRR and LBA produce similar results in most categories in MAD and RMSE. Both models produce reasonably good hedge result. In terms of index point error, CRR seems to slightly outperform LBA in many categories as well as the overall category in Table 7, although there are few cases where the LBA performs slightly better. In terms of percentage, CRR performs better with out-of-the-money calls and LBA performs better with in-the-money calls.

Implications. It is surprising that although LBA outperforms CRR in both the in-sample test and the out-of-sample test, their hedging performance is close,

**Table 7:** Dynamic Intrady Hedging Test Result

The dynamic intraday hedging test result is split into two panels. The index point hedging error is defined as $\delta$ actual option price $-\delta$ model option price, while the percentage hedging error is the index point hedging error as a percentage of model option price. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data of the pervious trading day.

Panel A: Index Points Error

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| $< 0.98$ | | | | | | | | |
| Mean | -0.002 | -0.011 | -0.014 | -0.015 | -0.051 | -0.036 | -0.005 | -0.012 |
| MAE | 0.073 | 0.089 | 0.064 | 0.064 | 0.052 | 0.038 | 0.071 | 0.084 |
| RMSE | 0.095 | 0.120 | 0.091 | 0.090 | 0.074 | 0.048 | 0.094 | 0.115 |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | 0.013 | 0.014 | 0.000 | -0.002 | -0.007 | -0.008 | 0.002 | 0.002 |
| MAE | 0.083 | 0.100 | 0.074 | 0.075 | 0.062 | 0.059 | 0.074 | 0.078 |
| RMSE | 0.107 | 0.130 | 0.093 | 0.095 | 0.080 | 0.077 | 0.094 | 0.103 |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | -0.002 | 0.009 | -0.003 | -0.007 | 0.002 | 0.001 | -0.001 | -0.001 |
| MAE | 0.096 | 0.112 | 0.073 | 0.081 | 0.092 | 0.097 | 0.084 | 0.093 |
| RMSE | 0.123 | 0.143 | 0.091 | 0.102 | 0.115 | 0.121 | 0.107 | 0.118 |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 0.002 | -0.009 | 0.004 | 0.009 | 0.005 | 0.008 | 0.004 | 0.006 |
| MAE | 0.098 | 0.118 | 0.100 | 0.114 | 0.122 | 0.139 | 0.108 | 0.124 |
| RMSE | 0.148 | 0.170 | 0.127 | 0.145 | 0.151 | 0.171 | 0.139 | 0.158 |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | -0.027 | 0.001 | -0.004 | -0.004 | -0.010 | -0.003 | -0.011 | -0.002 |
| MAE | 0.147 | 0.146 | 0.140 | 0.145 | 0.145 | 0.147 | 0.143 | 0.146 |
| RMSE | 0.216 | 0.227 | 0.184 | 0.194 | 0.183 | 0.183 | 0.193 | 0.200 |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 0.020 | 0.021 | 0.034 | 0.035 | 0.023 | 0.024 |
| MAE | NaN | NaN | 0.141 | 0.144 | 0.123 | 0.122 | 0.138 | 0.141 |
| RMSE | NaN | NaN | 0.192 | 0.194 | 0.161 | 0.161 | 0.188 | 0.189 |
| All Moneyness | | | | | | | | |
| Mean | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| MAE | 0.086 | 0.102 | 0.088 | 0.094 | 0.094 | 0.099 | 0.089 | 0.098 |
| RMSE | 0.117 | 0.138 | 0.117 | 0.126 | 0.122 | 0.129 | 0.118 | 0.131 |

Panel B: Percentage Error

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $< 0.98$ | | | | | | | | |
| Mean | -0.25% | -0.72% | -1.28% | -1.50% | -10.01% | -7.82% | -0.56% | -0.95% |
| MAE | 4.01% | 5.29% | 5.09% | 5.93% | 10.26% | 8.17% | 4.28% | 5.43% |
| RMSE | 5.38% | 7.03% | 7.06% | 8.14% | 15.55% | 10.93% | 5.96% | 7.29% |
| $0.98 - 0.99$ | | | | | | | | |
| Mean | 0.39% | 0.39% | -0.02% | -0.12% | -0.74% | -0.91% | -0.09% | -0.18% |
| MAE | 2.75% | 3.30% | 3.73% | 4.06% | 6.03% | 6.37% | 4.06% | 4.47% |
| RMSE | 3.57% | 4.29% | 4.66% | 5.09% | 7.74% | 8.15% | 5.44% | 5.90% |
| $0.99 - 1.00$ | | | | | | | | |
| Mean | -0.09% | 0.10% | -0.07% | -0.17% | 0.15% | 0.11% | 0.00% | -0.02% |
| MAE | 2.09% | 2.37% | 2.07% | 2.37% | 3.89% | 4.22% | 2.72% | 3.03% |
| RMSE | 2.73% | 3.03% | 2.61% | 2.99% | 5.10% | 5.52% | 3.72% | 4.09% |
| $1.00 - 1.01$ | | | | | | | | |
| Mean | 0.01% | -0.18% | 0.06% | 0.12% | 0.09% | 0.10% | 0.07% | 0.07% |
| MAE | 1.49% | 1.70% | 1.72% | 1.97% | 2.62% | 2.90% | 2.02% | 2.27% |
| RMSE | 2.28% | 2.50% | 2.17% | 2.54% | 3.27% | 3.56% | 2.63% | 2.95% |
| $1.01 - 1.02$ | | | | | | | | |
| Mean | -0.30% | 0.00% | 0.00% | 0.01% | -0.12% | -0.01% | -0.10% | 0.00% |
| MAE | 1.52% | 1.41% | 1.61% | 1.59% | 1.86% | 1.80% | 1.64% | 1.59% |
| RMSE | 2.27% | 2.22% | 2.15% | 2.15% | 2.37% | 2.27% | 2.23% | 2.19% |
| $\geq 1.02$ | | | | | | | | |
| Mean | NaN | NaN | 0.16% | 0.16% | 0.28% | 0.27% | 0.18% | 0.18% |
| MAE | NaN | NaN | 1.12% | 1.10% | 1.09% | 1.04% | 1.11% | 1.09% |
| RMSE | NaN | NaN | 1.51% | 1.46% | 1.42% | 1.36% | 1.49% | 1.44% |
| All Moneyness | | | | | | | | |
| Mean | -0.03% | -0.17% | -0.08% | -0.13% | -0.19% | -0.23% | -0.09% | -0.17% |
| MAE | 2.95% | 3.67% | 2.48% | 2.77% | 4.07% | 4.34% | 3.06% | 3.48% |
| RMSE | 4.13% | 5.20% | 3.47% | 3.88% | 5.64% | 5.90% | 4.36% | 4.92% |

and CRR even slightly outperforms LBA in some categories. The result may imply that as long as the hedging is rebalanced frequently enough, the hedging effectiveness will not be very sensitive to model misspecifications.

#### 4.3.4 Profitability Test

It is interesting to know whether the models can produce abnormal profits. The dynamic intraday hedge will not be feasible in a real world due to transaction costs and monitoring costs, etc. The daily rebalanced hedging is more realistic for the true purpose for testing the profitability of the models.

Consider the traditional delta-neutral hedge. For the CRR model, the only source of uncertainty arises from the underlying asset price changes, so a delta-neutral hedge portfolio is constructed using the underlying asset and one option. However, when volatility is stochastic, two options together with the underlying asset are needed to fully hedge the risks. To ensure that the hedging performance is measured correctly, we could select samples where both options as well as the underlying asset were traded concurrently in two consecutive days. However, this would lead to a very small sample and insignificant test results. It is also not practicable because the additional option required may not exist or does not trade frequently.

Alternatively, we can use the underlying asset to hedge each option to achieve minimum variance in the hedged portfolio value. This Minimum Variance Hedging approach is also common in the literature[7]. The minimum variance hedge ratio $x$ is given by

$$x = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial V}\frac{\rho b}{S}. \tag{11}$$

The derivation of the minimum variance hedge ratio is shown in Appendix B. Minimum variance portfolios are constructed for both models to assess their ability to produce profits. The minimum variance hedge ratio is calculated for every option in day $t$ using parameters implied through day $t-1$'s option prices. The hedged portfolio consists of $x$ short positions in the underlying futures, that requires zero initial cash outlay, and one long position in the option financed through shorting the risk-free asset. Thus, it is a zero-cost portfolio at day $t$. The portfolio is liquidated on day $t+1$, so the investor obtains the option value at day $t$, the difference of the futures prices of the two days, and covers the short position of the risk-free asset. A detailed representation of the process is shown in Table 8.

The hedge is repeated for each traded price on each day as long as there is transaction of the contract on the following day. This method effectively rolls over the hedged portfolio daily. Whether to long this zero-cost portfolio or to short it is decided by checking whether model call price is greater or less than the actual price. The hedged portfolio is valued at the last transaction price of day $t+1$ of the same contract. The profits are then measured by the average portfolio values. The profit is reported for different categories. The simple Student T-test is used to test for the significance of the profits.

[7] See Nandi (1996)[25] and Bakshi et al (1997)[2]

**Table 8:** Cash flow of the hedged portfolio

| Time | Call | Futures | Risk-free Asset | Value |
|---|---|---|---|---|
| $t$ | $-C_t$ | $x \cdot 0$ | $+C_t$ | $0$ |
| $t+1$ | $C_{t+1}$ | $x \cdot (F_t - F_{t+1})$ | $-C_t(1+r/365)$ | $C_{t+1} + x \cdot (F_t - F_{t+1}) - C_t \cdot (1+r/365)$ |

† $C_t$ is the call price on day $t$, $x$ is the hedge ratio, $F_t$ is the futures price on day $t$, and $r$ is the annualized risk-free rate.

**Table 9:** Minimum variance hedge ratios

The minimum hedge ratio for CRR is $\frac{\partial C}{\partial S}$ only, while for LBA, both $\frac{\partial C}{\partial S}$ and $\frac{\partial C}{\partial V}$, as shown in equation (11).

Panel A: Index Points Error

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| $< 0.98$ | | | | | | | | |
| Hedge Ratio | 0.212 | 0.109 | 0.190 | 0.105 | 0.133 | 0.076 | 0.208 | 0.108 |
| $\frac{\partial C}{\partial S}$ | 0.212 | 0.232 | 0.190 | 0.179 | 0.133 | 0.125 | 0.208 | 0.222 |
| $\frac{\partial C}{\partial V}$ | | 167.986 | | 139.669 | | 71.953 | | 162.430 |
| $0.98 - 0.99$ | | | | | | | | |
| Hedge Ratio | 0.317 | 0.205 | 0.279 | 0.186 | 0.219 | 0.131 | 0.275 | 0.177 |
| $\frac{\partial C}{\partial S}$ | 0.317 | 0.361 | 0.279 | 0.281 | 0.219 | 0.207 | 0.275 | 0.288 |
| $\frac{\partial C}{\partial V}$ | | 227.339 | | 178.433 | | 100.212 | | 173.250 |
| $0.99 - 1.00$ | | | | | | | | |
| Hedge Ratio | 0.433 | 0.331 | 0.421 | 0.340 | 0.405 | 0.316 | 0.418 | 0.330 |
| $\frac{\partial C}{\partial S}$ | 0.433 | 0.504 | 0.421 | 0.456 | 0.405 | 0.430 | 0.418 | 0.457 |
| $\frac{\partial C}{\partial V}$ | | 260.002 | | 217.288 | | 147.739 | | 201.708 |
| $1.00 - 1.01$ | | | | | | | | |
| Hedge Ratio | 0.565 | 0.484 | 0.577 | 0.526 | 0.600 | 0.555 | 0.584 | 0.530 |
| $\frac{\partial C}{\partial S}$ | 0.565 | 0.640 | 0.577 | 0.640 | 0.600 | 0.667 | 0.584 | 0.650 |
| $\frac{\partial C}{\partial V}$ | | 257.346 | | 211.454 | | 147.383 | | 194.718 |
| $1.01 - 1.02$ | | | | | | | | |
| Hedge Ratio | 0.694 | 0.610 | 0.708 | 0.656 | 0.786 | 0.733 | 0.722 | 0.662 |
| $\frac{\partial C}{\partial S}$ | 0.694 | 0.746 | 0.708 | 0.752 | 0.786 | 0.811 | 0.722 | 0.764 |
| $\frac{\partial C}{\partial V}$ | | 229.540 | | 181.381 | | 103.003 | | 176.184 |
| $\geq 1.02$ | | | | | | | | |
| Hedge Ratio | NaN | NaN | 0.837 | 0.777 | 0.875 | 0.815 | 0.844 | 0.784 |
| $\frac{\partial C}{\partial S}$ | NaN | NaN | 0.837 | 0.848 | 0.875 | 0.873 | 0.844 | 0.852 |
| $\frac{\partial C}{\partial V}$ | | NaN | | 130.042 | | 90.639 | | 123.037 |
| All Moneyness | | | | | | | | |
| Hedge Ratio | 0.335 | 0.233 | 0.456 | 0.381 | 0.429 | 0.354 | 0.410 | 0.326 |
| $\frac{\partial C}{\partial S}$ | 0.335 | 0.378 | 0.456 | 0.485 | 0.429 | 0.454 | 0.410 | 0.442 |
| $\frac{\partial C}{\partial V}$ | | 212.676 | | 193.620 | | 131.219 | | 183.010 |

Hedge Ratios. The minimum variance hedge ratios of both models are shown in Table 9. We find that $\frac{\partial C}{\partial S}$ of CRR is generally smaller than that of the LBA. However, the relative large $\frac{\partial C}{\partial V}$ and the negative correlation coefficient make the actual hedge ratio of the LBA smaller than that of the CRR in every case. In other words, the CRR model over-hedges when there is negative correlation between the volatility and the underlying asset. In addition, we find that in the 2795 minimum variance hedged portfolios constructed, LBA takes long positions in 1318 portfolios and short positions in the remaining 1477. This is quite evenly distributed because the pricing error between the LBA model and market price is evenly distributed around zero. On the contrary, CRR takes 2484 long positions and 311 short positions. This is partly due to the fact that CRR tends to overprice out-of-the-money options and our sample have more out- of-the-money calls, which is typical with American options since in-the-money options are

**Table 10:** Profitability Test Result

The profit test result is shown as the index points profit and percentage profit in each subgroup as well as subtotals and grand totals. Model prices are calculated using the estimated parameters using the sample data of the pervious trading day. If an actual option price is higher than theoretical price, i.e. the model price, then a zero-cost, delta-neutral portfolio is constructed to short the option, as in Table 4.3.4, and vice versa The profit is then defined as the portfolio value at the next day closing prices. Percentage profit is not based on investment cost, because that is zero. Instead, we use the option price when we construct the protfolio as 100%.

| | $T \geq 15$ | | $35 > T > 15$ | | $15 \geq T$ | | All Maturity | |
|---|---|---|---|---|---|---|---|---|
| Moneyness $\frac{S}{X}$ | CRR | LBA | CRR | LBA | CRR | LBA | CRR | LBA |
| $< 0.98$ | | | | | | | | |
| Index Point | -0.028* | 0.007 | -0.075* | 0.025 | -0.057 | 0.090 | -0.035* | 0.011 |
| Percentage | -1.74%* | 1.14% | -6.06%* | 0.82% | -8.11% | 14.65% | -2.47%* | 1.28% |
| $0.98 - 0.99$ | | | | | | | | |
| Index Point | 0.067* | 0.150* | -0.060* | 0.020 | -0.037 | 0.067* | -0.011 | 0.076* |
| Percentage | 2.83%* | 5.47%* | -3.68%* | 1.23% | -3.24% | 8.14% | -1.39% | 4.54%* |
| $0.99 - 1.00$ | | | | | | | | |
| Index Point | 0.059* | 0.069 | -0.023 | 0.072* | -0.127* | -0.052* | -0.042* | 0.027* |
| Percentage | 1.39%* | 1.49% | -0.91%* | 2.14%* | -5.34%* | -2.39%* | -1.99%* | 0.40% |
| $1.00 - 1.01$ | | | | | | | | |
| Index Point | -0.012 | 0.084 | -0.035 | 0.080* | -0.086* | 0.042 | -0.050* | 0.067* |
| Percentage | -0.20% | 1.11% | -0.63% | 1.57%* | -1.83%* | 0.97%* | -1.01%* | 1.29%* |
| $1.01 - 1.02$ | | | | | | | | |
| Index Point | -0.025 | 0.025 | -0.048 | -0.022 | -0.003 | 0.080 | -0.032 | 0.012 |
| Percentage | -0.22% | 0.21% | -0.60% | -0.34% | 0.05% | 0.93% | -0.36% | 0.08% |
| $\geq 1.02$ | | | | | | | | |
| Index Point | NaN | NaN | 0.000 | 0.098* | 0.039 | 0.126 | 0.007 | 0.103 |
| Percentage | NaN | NaN | 0.00% | 0.67% | 0.40% | 1.07% | 0.07% | 0.74%* |
| All Moneyness | | | | | | | | |
| Index Point | 0.017 | 0.065* | -0.038* | 0.053* | -0.083* | 0.013 | -0.032* | 0.046* |
| Percentage | 0.32% | 2.32%* | -1.74%* | 1.43%* | -3.59%* | 1.55% | -1.57%* | 1.75%* |

sometimes exercised before the maturity and out-of-the-money options always remain in the market.

Profitability. The profitability test result in Table 10 shows that the LBA outperforms the CRR in every category. LBA produces profit in all categories except for two, while the CRR produces loss in most cases. In terms of index points, LBA results in profits in all cases except for two. For options with moneyness between 1.01–1.02 while time-to-maturity between 15 and 35 days, LBA produces an insignificant loss of -0.022; and for options with less than 15 days to maturity and with moneyness between 0.99 and 1.00, where it produces a statistically significant loss. There are 11 cases out of the 28 categories, where the profits produced by LBA are statistically significant at 1%. CRR produces two significant profits for options with $T \geq 35$ and $S/X$ of 0.98–0.99 and 0.99–1.00. However, there are 11 categories where CRR produces significant losses. On the whole, LBA produces a significant profit of 0.046 while CRR produces a significant loss of -0.032.

The findings are similar in terms of percentage profits. LBA produces 11 significant profits and 1 significant loss, while CRR produces 12 significant losses and 2 significant profits. There is some inconsistency between the index point and percentage measure here as well. For example, the LBA produces a significant 0.067 profit for options with $T \geq 15$ and $S/X$ between 0.98–0.99, but at the same time, an insignificant 8.14% profit for the same category.

The findings here for the CRR model are consistent with Chang, Chang and Lim (1998)[9], who report negative profits for the Black and Scholes model with S&P 500 futures options during the period June 1994 to December 1994. The result is in contrast with Whaley (1986)[31], who reports significant profits. Chang, Chang and Lim (1998)[9] explain that the Black and Scholes model's ability of capturing abnormal profits may have been absorbed by the market over the years. Our test with more recent data verifies this point again, though our model shows smaller loss for CRR than theirs for the Black-Scholes model with quadratic approximation for the early exercise premiums.

Implications. The profits for LBA and losses for CRR show that LBA can actually predict price accurately and is even able to capture some opportunities for abnormal returns. On the other hand, CRR has pricing bias and is not able to produce any profits. However, the trading profits from LBA are not large enough to cover the transaction costs and the bid-ask spread for normal investors.

# 5 Conclusions

In this paper, we improve the lattice-based approach to numerically price American options under stochastic volatility. We extend the model to allow for mean reversion in the volatility process, correlation between the volatility and the underlying as well as non-zero risk premium of the volatility. The risk-neutral probability and jumps measure derived in this paper is very general and is easily adaptable to other processes. The model is computationally efficient, accurate and relatively easy to implement.

We empirically test the LBA model with S&P 500 futures options intraday data. Compared with traditional constant volatility model, LBA shows better in-sample and out-of-sample test results. The dynamic intraday hedging test shows that there is not much improvement in terms of hedging. However, the profitability test shows that there are opportunities for abnormal profit when trading with the LBA model. The profits are significant at 1% level, although the 0.046 index points or 1.75% profit will disappear when transaction costs are taken into considerations. The CRR model produces losses in this test.

The model can be improved by using better parameter estimation procedures, more appropriate interpolation and extrapolation methods. These are essentially calibration problems common in numerical option pricing. In addition, the model can be extended to include more factors such as stochastic interest rate and jumps.

# Appendix A The Derivation of the Risk-neutral Probabilities and Jump Sizes

Assuming the underlying asset follows a lognormal process and the squared volatility follows the squared root process as shown in equation (1) and (2). Further more, assume that at any point of time, both the underlying asset price $S$ and the squared volatility $V$ are followed by either an upward jump or a downward jump (discrete time price movement).

$$S^{+}_{i+1} = S_t + u_S \quad S^{-}_{i+1} = S_t + d_S \quad V^{+}_{i+1} = V_t + u_V \quad V^{-}_{i+1} = V_t + d_V, \tag{A-1}$$

where $u_S$, $d_S$, $u_V$, $d_V$ are discrete time additive up and down jumps of $S$ and $V$.

We use $p$ to denote the probability of an upward jump of $S$ occurs, and $q$ to denote the probability of an upward jump of $V$ occurs. $p_{11}$, $p_{12}$, $p21$, $p22$ are the joint probabilities of the respective jumps with the first subscript representing $S$, second for $V$, 1 for upward jump, and 2 for a downward jump. Then we have the following constrains for discrete jumps convergence to continuous process as $\Delta t \to 0$ by matching the first moments and second moments of the processes:

$$\begin{aligned}
&E(\Delta S) &:&\quad Sr\Delta t = pu_S + (1-p)d_S \\
&E(\Delta V) &:&\quad \kappa(\theta - V)\Delta t = qu_V + (1-q)d_V \\
&E(S^2) - [E(S)]^2 &:&\quad S^2V\Delta t = [pu_S^2 + (1-p)d_S^2] - [pu_S + (1-p)d_S]^2 \\
&E(V^2) - [E(V)]^2 &:&\quad b^2V\Delta t = [qu_V^2 + (1-q)d_V^2] - [qu_V + (1-q)d_V]^2 \\
&E(SV) - E(S)E(V) &:&\quad \rho SbV\Delta t = (p_{11}u_Su_V + p_{12}u_Sd_V + p_{21}d_Su_V + p_{22}d_Sd_V) \\
& & &\qquad - [pu_S + (1-p)d_S][qu_V + (1-q)d_V].
\end{aligned} \tag{A-2}$$

We also have some further constraints on the probabilities:

$$\begin{aligned}
p_{11} + p_{12} &= p \\
p_{11} + p_{12} &= q \\
p_{11} + p_{12} + p_{21} + p_{22} &= 1 \\
p = q &= 0.5.
\end{aligned} \tag{A-3}$$

The last condition is arbitrary, since we have more unknowns than constraints. We can also assume $u_S = -d_S$ and $u_V = -d_V$, similar to the Cox, Ross and Rubinstein's standard binomial tree setup. However, since the binomial tree is not recombining with stochastic volatility anyway, we prefer to set $p = q = 0.5$ to ensure that the probabilities are well defined. The solution to the above problem is equation (6).

# Appendix B The Derivation of Minimum Hedge Ratio

The derivation of minimum hedge ratio $x$ is as follows. It is assumed that the underlying asset and volatility process is given by equation (1) and eqrefe:dv.

A contingent claim on the underlying satisfies the following PDE:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial V}dV + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}dV^2 + \frac{\partial^2 C}{\partial S\,\partial V}dS\,dV. \tag{B-1}$$

Applying Ito's Lemma and ignoring higher orders of $dt$, the variance and covariance of the derivatives are given by:

$$\begin{aligned} dS^2 &= S^2 V dt \\ dV^2 &= b^2 V dt \\ dS\,dV &= \rho S b V dt \\ dC^2 &= \left(\left(\frac{\partial C}{\partial S}\right)^2 S^2 V + \left(\frac{\partial C}{\partial V}\right)^2 b^2 V + 2\frac{\partial C}{\partial S}\frac{\partial C}{\partial V}\rho S b V\right) dt \\ dC\,dS &= \left(\frac{\partial C}{\partial S}S^2 V + \frac{\partial C}{\partial V}\rho S b V\right) dt. \end{aligned} \tag{B-2}$$

If we construct a portfolio by longing one option and shorting $x$ underlying, the variance is:

$$\begin{aligned} Var(dC - xdS) &= Var(dC) + x^2 Var(dS) - 2xCov(dC, dS) \\ &= \left(\left(\frac{\partial C}{\partial S}\right)^2 S^2 V + \left(\frac{\partial C}{\partial V}^2 b^2 V\right) + 2\frac{\partial C}{\partial S}\frac{\partial C}{\partial V}\rho S b V\right) dt + x^2 S^2 V dt \\ &\quad - 2x\left(\frac{\partial C}{\partial S}S^2 V + \frac{\partial C}{\partial V}\rho S b V\right) dt. \end{aligned} \tag{B-3}$$

The first order condition is:

$$\frac{\partial Var(dC - xdS)}{\partial x} = 2xS^2 V dt - 2\left(\frac{\partial C}{\partial S}S^2 V + \frac{\partial C}{\partial V}\rho S b V\right) dt = 0. \tag{B-4}$$

Solving for $x$ from equation (B-4), we obtain the solution of $x$ in equation (11).

If we use opposite positions in the risk-free asset to finance the portfolio, the above solution is also obtained when we assume constant interest rate, or deterministic interest rate that has zero correlation with the futures and options. The minimum variance hedge ratio for a constant volatility model is simply the first term in the above equation since the correlation coefficient $\rho$ is zero. So for constant volatility model, the minimum variance hedge is the same as the traditional delta-neutral hedge.

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