The Variance Ratio Test with Stable Paretian Errors

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Abstract: In this paper we examine the distribution of the variance ratio statistic when the errors are distributed with thick tails as described by the family of stable Paretian distributions. The asymptotic distribution of the OVR statistic, which depends on the characteristic exponent, can be estimated using simulation. It is found that the convergence of the distribution of the OVR statistic to its asymptotic limit is extremely slow. Thus, the asymptotic result will not be able to provide any useful approximation in finite samples. To facilitate the OVR statistic as a test for the random walk hypothesis, we estimated the tail quantiles for several finite sample sizes.

Key Words: Monte Carlo experiment, random walk hypothesis, stable Paretian distribution, variance ratio test

JEL Classification: C12

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1 Introduction

Since the earlier works of Cochrane (1988) and Lo and MacKinlay (1988, 1989) the variance ratio statistic has been used widely as a test for the random walk hypothesis. For applications of the test see, for example, Campbell and Mankiw (1987) and Poterba and Summers (1988). Cecchetti and Lam (1994) studied the small-sample properties of the test.

Lo and MacKinlay (1988) provided the asymptotic theory for the variance ratio test. Their results, however, are based on Gaussian assumption. It is well known in the finance literature that many asset returns are distributed with heavy tails, which is inconsistent with the Gaussian assumption. In this paper we examine the distribution of the variance ratio statistic when the errors are distributed with thick tails as described by the family of stable Paretian distributions. The results due to Davis and Resnick (1986) are used to obtain the asymptotic distribution of the variance ratio statistic. These results have been used by Phillips and Loretan (1991) to study the Durbin-Watson statistic when the errors have infinite variance.

The plan of this paper is as follows. In Section 2 we briefly review the use of the variance ratio statistic. Section 3 defines our notations for the stable Paretian distribution. The estimation of the asymptotic distribution of the variance ratio statistic is discussed in Section 4. It is found that the rate of convergence of the distribution of the variance ratio to its asymptotic limit is very slow. To facilitate the use of the variance ratio statistic as a test for the random walk hypothesis in finite samples we estimate the quantiles of the statistic using a large scale Monte Carlo experiment. These results are presented in Section 5. Section 6 provides some concluding remarks.

2 The Overlapping Variance Ratio Test

Let $\{X_t\}$ denote a time series generated from the following equation

$$X_t = \mu + X_{t-1} + \varepsilon_t. \tag{1}$$

We denote H_0 as the null hypothesis that ε_t are independently and identically distributed (IID) as Gaussian variates with mean zero and variance σ^2. That is,

$$H_0 : \varepsilon_t \sim \text{IID } N(0, \sigma^2).$$

Assuming the data consist of $nq + 1$ observations $X_0, X_1, \ldots, X_{nq}$, where n and q are arbitrary integers, we define

$$\hat{\mu} = \frac{1}{nq}\sum_{t=1}^{nq}(X_t - X_{t-1}) \tag{2}$$

and

$$\hat{\sigma}^2 = \frac{1}{nq}\sum_{t=1}^{nq}(X_t - X_{t-1} - \hat{\mu})^2. \tag{3}$$

Note that $\hat{\sigma}^2$ is the sample variance of the 1-period difference of X_t, namely, $\Delta X_t = X_t - X_{t-1}$. This is the maximum likelihood estimator (MLE) of the parameter σ^2. Alternatively we may consider the q-period difference of X_t, $X_t - X_{t-q}$, from which we obtain an estimate of the variance of the q-period difference as

$$\frac{1}{nq}\sum_{t=q}^{nq}(X_t - X_{t-q} - q\hat{\mu})^2.$$

Under H_0, the variance of the q-period difference of X_t is $q\sigma^2$. Thus, an asymptotically unbiased estimate of σ^2 can be obtained by defining $\hat{\sigma}_q^2$ as

$$\hat{\sigma}_q^2 = \frac{1}{nq^2}\sum_{t=q}^{nq}(X_t - X_{t-q} - q\hat{\mu})^2. \tag{4}$$

A test for the random walk hypothesis H_0 may be constructed by considering the centered ratio

$$M_q = \frac{\hat{\sigma}_q^2}{\hat{\sigma}^2} - 1. \tag{5}$$

Lo and MacKinlay (1988) showed that under H_0,

$$\sqrt{nq}\,M_q \xrightarrow{D} N\left(0, \frac{2(2q-1)(q-1)}{3q}\right), \tag{6}$$

where $\xrightarrow{D}$ denotes convergence in distribution as $n \to \infty$ keeping q fixed. Defining

$$V_q = \frac{2(2q-1)(q-1)}{3nq^2}, \tag{7}$$

the overlapping variance ratio (OVR) statistic can be calculated as[1]

$$R_q = \frac{M_q}{\sqrt{V_q}}, \tag{8}$$

which is asymptotically distributed as $N(0, 1)$ under H_0. Lo and MacKinlay (1989) reported some results on the finite-sample distributions of the OVR statistic R_q. They discussed the relationship between the OVR test as a test for the random walk hypothesis and various tests for the unit-root hypothesis. They argued that the OVR test is preferred when the attribute of interest is the uncorrelatedness of the increments. Also, the independence of the OVR statistic of any nuisance parameters is an advantage over the regression-based tests for the unit root hypothesis.

It should be noted that M_q is closely related to the sample autocorrelation coefficients. Defining $\hat{\rho}_i$ as

$$\hat{\rho}_i = \frac{\sum_{t=i+1}^{nq}(\Delta X_t - \hat{\mu})(\Delta X_{t-i} - \hat{\mu})}{\sum_{t=1}^{nq}(\Delta X_t - \hat{\mu})^2}, \tag{9}$$

it can be shown that (see Lo and MacKinlay (1989)), for $q > 1$,

$$M_q = \frac{2(q-1)}{q}\hat{\rho}_1 + \frac{2(q-2)}{q}\hat{\rho}_2 + \cdots + \frac{2}{q}\hat{\rho}_{q-1} + o_p(n^{-1/2}). \tag{10}$$

Thus, the OVR statistic is approximately a linear combination of the first $q-1$ sample autocorrelation coefficients of the 1-period difference with declining weights. This result will be used below to obtain the asymptotic distribution of the OVR statistic under stable Paretian errors.

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[1] A nonoverlapping variance ratio (NVR) statistic can be calculated by considering $\bar{\sigma}_q^2 = [\sum_{t=1}^{n}(X_{tq} - X_{(t-1)q} - q\hat{\mu})^2]/(nq)$ in place of $\hat{\sigma}_q^2$ in M_q. As pointed out by Campbell, Lo and MacKinlay (1997) the OVR statistic provides an asymptotically more powerful test than the NVR statistic. In this paper we shall consider the OVR statistic only.

3 The Stable Paretian Distribution

While the asymptotic theory of the OVR statistic as discussed in the last section was developed based on the assumption of Gaussian errors, it has been widely reported in the finance and economics literature that many financial and economic time series are distributed with thick tails. To incorporate these empirical findings we adopt the assumption that the errors of the increments are distributed as stable Paretian variates. In this section we define the notations used for the stable Paretian law.

There are several variations of definitions of a stable Paretian distribution. In this paper we follow the definition as found in Adler, Feldman and Gallagher (1998). Let Y be a stable Paretian variable with parameters $(\alpha; \sigma; \beta; \mu)$; denoted as $Y = S_\alpha(\sigma; \beta; \mu)$. The characteristic function of Y is given by

$$E(e^{itY}) = \begin{cases} \exp\left\{ i\mu t - \sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan\frac{\pi\alpha}{2}\right) \right\} & \alpha \neq 1 \\ \exp\left\{ i\mu t - \sigma|t| \left(1 + \frac{2}{\pi} i\beta \operatorname{sign}(t) \ln|t|\right) \right\} & \alpha = 1 \end{cases} \tag{11}$$

where $i^2 = -1$, α is the characteristic exponent parameter satisfying $0 < \alpha \leq 2$, $\sigma > 0$ is the scale parameter, β is the skewness parameter satisfying $-1 \leq \beta \leq 1$, and μ is the location parameter.

The characteristic exponent α relates directly to the heaviness of the tails of the stable Paretian distribution. The case of $\alpha = 2$ corresponds to a Gaussian distribution, while the case of $\alpha = 1$ corresponds to a Cauchy distribution. For these two special cases, closed-form expressions of the density functions exist. Otherwise, the method due to McCulloch (1998) may be used to approximate the stable Paretian density. In contrast with Gaussian distributions a stable Paretian distribution with a characteristic exponent less than 2 has no finite variance.[2]

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[2] Lo and MacKinlay (1988) discussed the case in which the errors are heteroscedastic and thus not IID Gaussian. Their asymptotic theory, however, assumes that the errors have finite variance.

4 Asymptotic Distribution of the OVR Statistic under Stable Paretian Errors

Brockwell and Davis (1991, chapter 13) discussed the properties of linear processes with infinite variance. The results below summarize some important properties of the autocorrelation function of such processes. Let $\{Y_t\}$ be an IID symmetric sequence (i.e., $\beta = 0$) of α-stable random variables and let $\{Z_t\}$ be the strictly stationary process defined by

$$Z_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}; \qquad t = 1;\ldots;T \tag{12}$$

where

$$\sum_{j=-\infty}^{\infty} |j||\psi_j|^{\delta} < \infty \ \text{ for some } \delta \in (0;\alpha) \cap [0;1]:$$

For such a process we define an analogue of the autocorrelation function as

$$\rho_h = \frac{\sum_j \psi_j \psi_{h+j}}{\sum_j \psi_j^2}; \qquad h = 1;2;\cdots: \tag{13}$$

Then, for any positive integer h,

$$\left(\frac{T}{\ln T}\right)^{1/\alpha} (\hat{\rho}_1 - \rho_1;\cdots;\hat{\rho}_h - \rho_h)' \xrightarrow{D} (W_1;\cdots;W_h)'; \tag{14}$$

where

$$W_k = \sum_{j=1}^{\infty} (\rho_{k+j} + \rho_{k-j} - 2\rho_j \rho_k)\frac{S_j}{S_0}; \qquad k = 1;\ldots;h;$$

and

$$\hat{\rho}_i = \frac{\sum_{t=i+1}^{T} Z_t Z_{t-i}}{\sum_{t=1}^{T} Z_t^2}: \tag{15}$$

Here $S_0;S_1;S_2;\cdots$ are independent stable Paretian variables. S_0 is positive with $S_0 \sim S_{\alpha/2}(\sigma_0;1;0)$, while S_j are IID $S_\alpha(\sigma_1;0;0)$; for $j = 1;\ldots$, where

$$\sigma_0 = \begin{cases} \left(\dfrac{(1-\alpha)\Gamma(1-\frac{\alpha}{2})\cos(\frac{\pi\alpha}{4})}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}\right)^{2/\alpha} & \text{if } \alpha \neq 1 \\ \left[\frac{2}{\pi}\Gamma(\frac{1}{2})\cos(\frac{\pi}{4})\right]^2 & \text{if } \alpha = 1 \end{cases} \tag{16}$$

$$\sigma_1 = \begin{cases} \left(\dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}\right)^{1/\alpha} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} & \text{if } \alpha = 1 \end{cases}$$

For $\alpha > 1$, the above results are also valid when $\tilde{\rho}_i$ are replaced by their mean-corrected version

$$\hat{\rho}_i = \frac{\sum_{t=i+1}^{T}(Z_t - \bar{Z})(Z_{t-i} - \bar{Z})}{\sum_{t=1}^{T}(Z_t - \bar{Z})^2}; \tag{17}$$

where $\bar{Z}$ is the sample mean of $Z_1; \ldots; Z_T$.

Under the null hypothesis that $Z_t \equiv \Delta X_t$ are IID symmetric α-stable Paretian distributions, we have $\rho_i = 0$ for $i \geq 1$ and $\rho_0 = 1$. Hence, for $\alpha > 1$,

$$\left(\frac{T}{\ln T}\right)^{1/\alpha}(\hat{\rho}_1; \cdots; \hat{\rho}_h)' \xrightarrow{D} (W_1; \cdots; W_h)'; \tag{18}$$

where W_k are given by

$$W_k = \frac{S_k}{S_0}; \qquad k = 1; \ldots; h. \tag{19}$$

The above results, together with equation (10), provide a method for calculating the asymptotic distribution of the OVR statistic under stable Paretian errors. Assuming $T = nq$, we generate stable Paretian variates $S_0; S_1; \ldots; S_{q-1};$ with parameter values given by equation (16). From these we calculate $W_1; \ldots; W_{q-1}$ from equation (19) and subsequently obtain

$$\left(\frac{T}{\ln T}\right)^{1/\alpha} M_q \approx \frac{2(q-1)}{q}W_1 + \frac{2(q-2)}{q}W_2 + \ldots + \frac{2}{q}W_{q-1}. \tag{20}$$

Thus, the right-hand-side of the above equation gives the asymptotic distribution of the normalized OVR statistic. From now on we shall refer to $(T/\ln T)^{1/\alpha} M_q$ as the normalized OVR statistic.

Chambers, Mallows and Stuck (1976) provided an efficient algorithm to generate stable Paretian variates. We used this algorithm to simulate the asymptotic distribution of the normalized OVR statistic using the above method. Figures 1, 2 and 3 show the asymptotic distribution functions of the normalized OVR statistics for $q = 2$, 4 and 6, respectively. The cases of $\alpha = 1.9$, 1.7 and 1.5 are presented. To estimate the empirical asymptotic distribution function Monte Carlo runs of 50000 samples of $S_0; S_1; \ldots; S_{q-1}$

were generated. It can be seen from these graphs that the asymptotic distributions have thicker tails when α decreases or q increases.[3]

To examine the performance of the asymptotic distribution as an approximation to finite samples, we calculated the normalized OVR statistics based on observations generated from a random walk with stable Paretian errors. Specifically, we generated observations from equation (1) with ε_t being IID $S_{\alpha}(1; 0; 0)$. We considered $q = 2\ (1)\ 6$, $\alpha = 1.9\ (-0.1)\ 1.0$ and $T = nq = 300$, 6000 and 30000. Figures 4 through 9 present the results for selected cases of the experiment. The empirical quantile function (EQF) of the normalized OVR statistic based on Monte Carlo runs of 5000 samples are compared against the estimated theoretical asymptotic distribution function. It can be seen that the finite-sample distribution differs a lot from the asymptotic distribution even for $T = 30000$. These results suggest that the asymptotic distribution is not likely to provide any useful approximation to the finite-sample distribution of any reasonable size encountered in practice.

The poor performance of the asymptotic distribution of the normalized OVR statistic as an approximation to finite samples is a consequence of the slow convergence of the sample autocorrelation function to the limiting distribution. This result has been reported by Adler, Feldman and Gallagher (1998). These authors also pointed out that the averages of stable Paretian distributions converge to their limiting distributions very slowly. The reason for this phenomenon, however, is unknown. While our results reinforce this finding, methods to improve the finite-sample approximations involving statistics of stable Paretian distributions remain an important topic for future research.

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[3] In the Monte Carlo experiment we consider $q = 2\ (1)\ 6$ and $\alpha = 1.9\ (-0.1)\ 1.0$. The regularity of the tail behaviour of the empirical asymptotic distribution applies to these values considered. To conserve space, only the selected graphs are presented. Note that the case of $\alpha = 1.0$ has been included in the simulation although the asymptotic distribution theory applies only to $\alpha > 1$.

5 Finite-Sample Distribution of the OVR Statistic under Stable Paretian Errors

To facilitate the use of the normalized OVR statistic as a test for the random walk hypothesis under stable Paretian errors we estimated the quantiles of the statistic in finite samples. We considered $T = 120$, 240, 600, 1200 and 2400, with $q = 2\ (1)\ 6$ and $\alpha = 1.9\ (-0.1)\ 1.0$. Based on Monte Carlo runs of 100000 samples we estimated the quantiles at 2.5%, 5.0%, 95% and 97.5%. The results are summarized in Tables 1 through 5. These tables provide estimates of the critical values for testing the random walk hypothesis in finite samples.

As the critical values depend on the parameter α, this value has to be estimated. In this respect, the methods due to McCulloch (1986) or Koutrouvelis (1980, 1981) can be used. The recent paper by Kogon and Williams (1998) suggests a new method and provides a comparison of the various estimation methods. In actual applications, the size of the test would depend on the estimated value of α. This problem, however, will not be pursued here.

6 Conclusions

We have examined the distribution of the OVR statistic when the errors follow a stable Paretian distribution. The asymptotic distribution of the OVR statistic, which depends on the characteristic exponent α, can be estimated using simulation. It was found, however, that the convergence of the distribution of the OVR statistic to its asymptotic limit is extremely slow. Thus, the asymptotic result will not be able to provide any useful approximation in finite samples. To facilitate the OVR statistic as a test for the random walk hypothesis, we estimated the tail quantiles for several finite sample sizes, which may be used for practical applications.

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Table 1. Finite-Sample Quantiles of the OVR Statistic: $T = 120$

®	%	q = 2	q = 3	q = 4	q = 5	q = 6
1:90	2.5	-1.053	-1.531	-1.886	-2.174	-2.430
	5.0	-0.894	-1.323	-1.650	-1.923	-2.165
	95.0	0.706	1.038	1.249	1.432	1.558
	97.5	0.862	1.283	1.585	1.850	2.026
1:80	2.5	-1.151	-1.677	-2.093	-2.410	-2.678
	5.0	-0.966	-1.439	-1.820	-2.125	-2.375
	95.0	0.754	1.098	1.336	1.528	1.666
	97.5	0.927	1.373	1.699	1.966	2.167
1:70	2.5	-1.284	-1.874	-2.332	-2.694	-2.996
	5.0	-1.060	-1.591	-2.008	-2.354	-2.649
	95.0	0.816	1.184	1.454	1.654	1.805
	97.5	1.021	1.489	1.864	2.154	2.362
1:60	2.5	-1.462	-2.124	-2.650	-3.064	-3.400
	5.0	-1.185	-1.784	-2.257	-2.647	-2.958
	95.0	0.897	1.279	1.581	1.820	1.958
	97.5	1.124	1.638	2.041	2.363	2.594
1:50	2.5	-1.661	-2.470	-3.077	-3.540	-3.935
	5.0	-1.324	-2.026	-2.577	-3.027	-3.390
	95.0	0.984	1.439	1.746	1.971	2.191
	97.5	1.267	1.863	2.284	2.613	2.887
1:40	2.5	-1.950	-2.931	-3.692	-4.174	-4.678
	5.0	-1.518	-2.355	-2.990	-3.524	-3.992
	95.0	1.107	1.591	1.955	2.191	2.412
	97.5	1.465	2.128	2.586	2.931	3.283
1:30	2.5	-2.375	-3.555	-4.515	-5.098	-5.684
	5.0	-1.765	-2.793	-3.582	-4.214	-4.771
	95.0	1.244	1.820	2.197	2.515	2.737
	97.5	1.674	2.443	2.991	3.439	3.745
1:20	2.5	-2.966	-4.498	-5.640	-6.457	-7.181
	5.0	-2.139	-3.397	-4.358	-5.232	-5.920
	95.0	1.437	2.131	2.548	2.918	3.217
	97.5	2.018	2.993	3.536	4.094	4.544
1:10	2.5	-3.818	-5.845	-7.269	-8.381	-9.265
	5.0	-2.653	-4.337	-5.505	-6.677	-7.592
	95.0	1.680	2.496	3.064	3.510	3.871
	97.5	2.471	3.598	4.388	5.041	5.578
1:00	2.5	-5.214	-7.842	-10.044	-11.547	-12.783
	5.0	-3.467	-5.661	-7.362	-8.993	-10.361
	95.0	2.024	3.056	3.790	4.366	4.820
	97.5	3.135	4.619	5.633	6.428	7.148

Table 2. Finite-Sample Quantiles of the OVR Statistic: T = 240

®	%	q = 2	q = 3	q = 4	q = 5	q = 6
1:90	2.5	-0.973	-1.432	-1.770	-2.079	-2.332
	5.0	-0.820	-1.229	-1.532	-1.809	-2.034
	95.0	0.701	1.018	1.272	1.459	1.628
	97.5	0.851	1.246	1.575	1.824	2.063
1:80	2.5	-1.094	-1.607	-2.003	-2.322	-2.615
	5.0	-0.906	-1.357	-1.704	-1.988	-2.262
	95.0	0.761	1.124	1.376	1.596	1.764
	97.5	0.937	1.402	1.727	2.000	2.226
1:70	2.5	-1.212	-1.818	-2.266	-2.668	-3.011
	5.0	-0.991	-1.514	-1.912	-2.265	-2.570
	95.0	0.833	1.222	1.501	1.734	1.948
	97.5	1.030	1.539	1.901	2.192	2.501
1:60	2.5	-1.380	-2.086	-2.582	-3.091	-3.466
	5.0	-1.114	-1.705	-2.145	-2.591	-2.923
	95.0	0.911	1.356	1.687	1.913	2.122
	97.5	1.148	1.718	2.159	2.474	2.750
1:50	2.5	-1.641	-2.463	-3.116	-3.658	-4.098
	5.0	-1.275	-1.958	-2.526	-2.979	-3.408
	95.0	1.025	1.508	1.867	2.163	2.417
	97.5	1.323	1.983	2.428	2.851	3.171
1:40	2.5	-1.945	-2.964	-3.790	-4.455	-5.012
	5.0	-1.471	-2.301	-2.968	-3.567	-4.057
	95.0	1.156	1.726	2.166	2.489	2.753
	97.5	1.548	2.308	2.855	3.329	3.703
1:30	2.5	-2.365	-3.718	-4.661	-5.621	-6.247
	5.0	-1.722	-2.784	-3.581	-4.361	-4.955
	95.0	1.322	1.999	2.476	2.909	3.238
	97.5	1.823	2.737	3.379	3.942	4.404
1:20	2.5	-3.001	-4.758	-6.091	-7.180	-8.216
	5.0	-2.108	-3.413	-4.546	-5.421	-6.300
	95.0	1.571	2.383	2.987	3.516	3.883
	97.5	2.249	3.376	4.205	4.880	5.438
1:10	2.5	-3.938	-6.346	-8.121	-9.897	-11.107
	5.0	-2.596	-4.385	-5.838	-7.183	-8.384
	95.0	1.868	2.875	3.661	4.271	4.828
	97.5	2.844	4.277	5.316	6.259	6.989
1:00	2.5	-5.510	-9.078	-11.527	-14.054	-15.819
	5.0	-3.397	-5.844	-8.056	-9.852	-11.526
	95.0	2.346	3.640	4.741	5.474	6.200
	97.5	3.735	5.667	7.184	8.371	9.273

Table 3. Finite-Sample Quantiles of the OVR Statistic: T = 600

®	%	q = 2	q = 3	q = 4	q = 5	q = 6
1:90	2.5	-0.903	-1.332	-1.661	-1.933	-2.186
	5.0	-0.753	-1.125	-1.418	-1.655	-1.875
	95.0	0.682	0.998	1.260	1.464	1.636
	97.5	0.823	1.213	1.546	1.797	2.021
1:80	2.5	-1.013	-1.498	-1.895	-2.213	-2.504
	5.0	-0.838	-1.245	-1.590	-1.875	-2.127
	95.0	0.746	1.125	1.377	1.610	1.799
	97.5	0.909	1.373	1.698	2.006	2.249
1:70	2.5	-1.161	-1.737	-2.205	-2.558	-2.905
	5.0	-0.941	-1.418	-1.818	-2.125	-2.431
	95.0	0.835	1.241	1.566	1.814	2.027
	97.5	1.037	1.543	1.953	2.283	2.554
1:60	2.5	-1.340	-2.037	-2.580	-3.064	-3.436
	5.0	-1.063	-1.636	-2.086	-2.480	-2.821
	95.0	0.934	1.394	1.749	2.042	2.323
	97.5	1.186	1.787	2.234	2.635	2.980
1:50	2.5	-1.615	-2.448	-3.130	-3.667	-4.210
	5.0	-1.226	-1.901	-2.451	-2.906	-3.360
	95.0	1.056	1.599	2.026	2.363	2.629
	97.5	1.403	2.093	2.666	3.077	3.433
1:40	2.5	-1.947	-2.992	-3.851	-4.585	-5.218
	5.0	-1.436	-2.263	-2.926	-3.509	-4.073
	95.0	1.226	1.864	2.346	2.808	3.119
	97.5	1.657	2.529	3.122	3.732	4.162
1:30	2.5	-2.389	-3.803	-4.966	-5.970	-6.822
	5.0	-1.692	-2.734	-3.622	-4.385	-5.091
	95.0	1.429	2.215	2.826	3.288	3.749
	97.5	2.000	3.080	3.899	4.545	5.162
1:20	2.5	-3.099	-4.952	-6.600	-7.920	-9.075
	5.0	-2.091	-3.383	-4.629	-5.555	-6.550
	95.0	1.687	2.647	3.447	4.064	4.594
	97.5	2.506	3.852	4.992	5.897	6.521
1:10	2.5	-4.107	-6.749	-9.072	-11.070	-12.862
	5.0	-2.596	-4.441	-6.050	-7.534	-8.877
	95.0	2.073	3.335	4.322	5.142	5.952
	97.5	3.264	5.105	6.381	7.743	8.841
1:00	2.5	-5.582	-9.993	-13.509	-16.480	-19.318
	5.0	-3.358	-6.063	-8.422	-10.680	-12.759
	95.0	2.623	4.357	5.704	6.969	8.172
	97.5	4.391	7.162	9.247	10.831	12.819

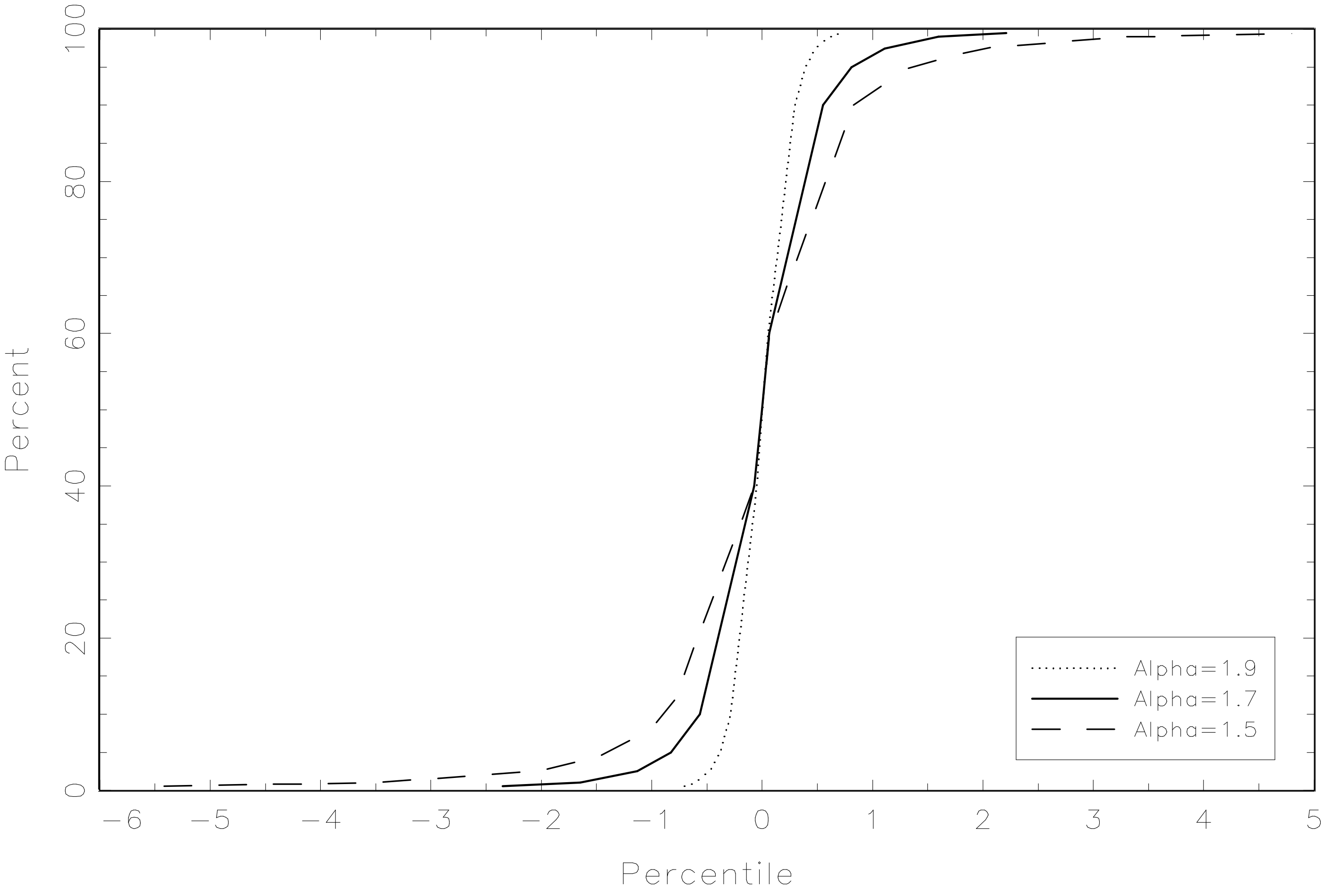
Table 4. Finite-Sample Quantiles of the OVR Statistic: T = 1200

®	%	q = 2	q = 3	q = 4	q = 5	q = 6
1:90	2.5	-0.851	-1.266	-1.588	-1.869	-2.091
	5.0	-0.711	-1.067	-1.339	-1.589	-1.783
	95.0	0.663	0.991	1.238	1.435	1.612
	97.5	0.798	1.197	1.504	1.753	1.977
1:80	2.5	-0.972	-1.460	-1.824	-2.127	-2.422
	5.0	-0.805	-1.211	-1.520	-1.791	-2.036
	95.0	0.737	1.099	1.392	1.615	1.822
	97.5	0.904	1.352	1.712	1.994	2.253
1:70	2.5	-1.118	-1.699	-2.150	-2.530	-2.852
	5.0	-0.902	-1.372	-1.756	-2.069	-2.356
	95.0	0.834	1.256	1.554	1.823	2.069
	97.5	1.035	1.570	1.944	2.293	2.597
1:60	2.5	-1.336	-1.989	-2.543	-3.001	-3.440
	5.0	-1.053	-1.593	-2.024	-2.406	-2.759
	95.0	0.948	1.415	1.793	2.122	2.392
	97.5	1.221	1.822	2.290	2.704	3.054
1:50	2.5	-1.603	-2.421	-3.163	-3.711	-4.249
	5.0	-1.203	-1.861	-2.426	-2.895	-3.324
	95.0	1.091	1.642	2.108	2.435	2.766
	97.5	1.437	2.173	2.754	3.174	3.617
1:40	2.5	-1.921	-2.992	-3.894	-4.673	-5.401
	5.0	-1.405	-2.226	-2.892	-3.493	-4.044
	95.0	1.266	1.946	2.512	2.926	3.363
	97.5	1.731	2.636	3.412	3.959	4.523
1:30	2.5	-2.409	-3.920	-5.178	-6.085	-7.077
	5.0	-1.685	-2.752	-3.688	-4.370	-5.095
	95.0	1.487	2.344	2.952	3.514	4.073
	97.5	2.129	3.302	4.185	4.988	5.733
1:20	2.5	-3.144	-5.064	-6.777	-8.304	-9.567
	5.0	-2.076	-3.454	-4.663	-5.714	-6.649
	95.0	1.821	2.919	3.734	4.547	5.158
	97.5	2.707	4.322	5.534	6.725	7.461
1:10	2.5	-4.187	-7.123	-9.485	-11.965	-13.987
	5.0	-2.627	-4.505	-6.139	-7.715	-9.194
	95.0	2.223	3.618	4.776	5.799	6.712
	97.5	3.572	5.638	7.464	8.898	10.312
1:00	2.5	-5.671	-10.628	-14.377	-17.975	-21.214
	5.0	-3.372	-6.286	-8.655	-10.941	-13.198
	95.0	2.781	4.671	6.598	7.914	9.360
	97.5	4.771	7.948	10.681	12.908	15.198

Table 5. Finite-Sample Quantiles of the OVR Statistic: T = 2400

®	%	q = 2	q = 3	q = 4	q = 5	q = 6
1:90	2.5	-0.822	-1.214	-1.526	-1.809	-2.031
	5.0	-0.682	-1.019	-1.285	-1.521	-1.709
	95.0	0.649	0.972	1.213	1.414	1.590
	97.5	0.779	1.172	1.476	1.712	1.931
1:80	2.5	-0.942	-1.411	-1.786	-2.098	-2.343
	5.0	-0.773	-1.167	-1.468	-1.743	-1.955
	95.0	0.738	1.106	1.377	1.616	1.816
	97.5	0.901	1.351	1.694	1.982	2.249
1:70	2.5	-1.107	-1.658	-2.105	-2.486	-2.830
	5.0	-0.889	-1.350	-1.717	-2.022	-2.307
	95.0	0.837	1.255	1.572	1.840	2.096
	97.5	1.049	1.571	1.967	2.299	2.630
1:60	2.5	-1.307	-2.004	-2.531	-2.974	-3.402
	5.0	-1.020	-1.574	-1.989	-2.360	-2.712
	95.0	0.952	1.447	1.825	2.134	2.449
	97.5	1.222	1.859	2.330	2.739	3.129
1:50	2.5	-1.594	-2.421	-3.176	-3.721	-4.277
	5.0	-1.182	-1.849	-2.413	-2.865	-3.303
	95.0	1.109	1.681	2.138	2.568	2.893
	97.5	1.479	2.250	2.838	3.348	3.827
1:40	2.5	-1.982	-3.053	-3.936	-4.727	-5.427
	5.0	-1.414	-2.234	-2.911	-3.475	-4.068
	95.0	1.284	1.997	2.589	3.068	3.530
	97.5	1.771	2.727	3.545	4.167	4.834
1:30	2.5	-2.460	-3.902	-5.207	-6.326	-7.429
	5.0	-1.692	-2.739	-3.649	-4.452	-5.220
	95.0	1.551	2.441	3.120	3.836	4.363
	97.5	2.256	3.551	4.498	5.477	6.157
1:20	2.5	-3.214	-5.355	-6.994	-8.619	-9.854
	5.0	-2.086	-3.503	-4.688	-5.824	-6.669
	95.0	1.845	3.026	4.011	4.877	5.645
	97.5	2.837	4.636	6.062	7.262	8.328
1:10	2.5	-4.247	-7.355	-9.965	-12.363	-14.830
	5.0	-2.622	-4.563	-6.312	-7.886	-9.412
	95.0	2.334	3.872	5.224	6.298	7.411
	97.5	3.809	6.230	8.282	9.993	11.649
1:00	2.5	-6.028	-10.768	-14.925	-18.628	-22.245
	5.0	-3.460	-6.388	-8.849	-11.183	-13.564
	95.0	2.932	5.162	7.035	8.828	10.358
	97.5	5.208	8.930	11.983	15.049	17.285

Figure 1: Asmptotic Distribution of the OVR Statistic: q=2



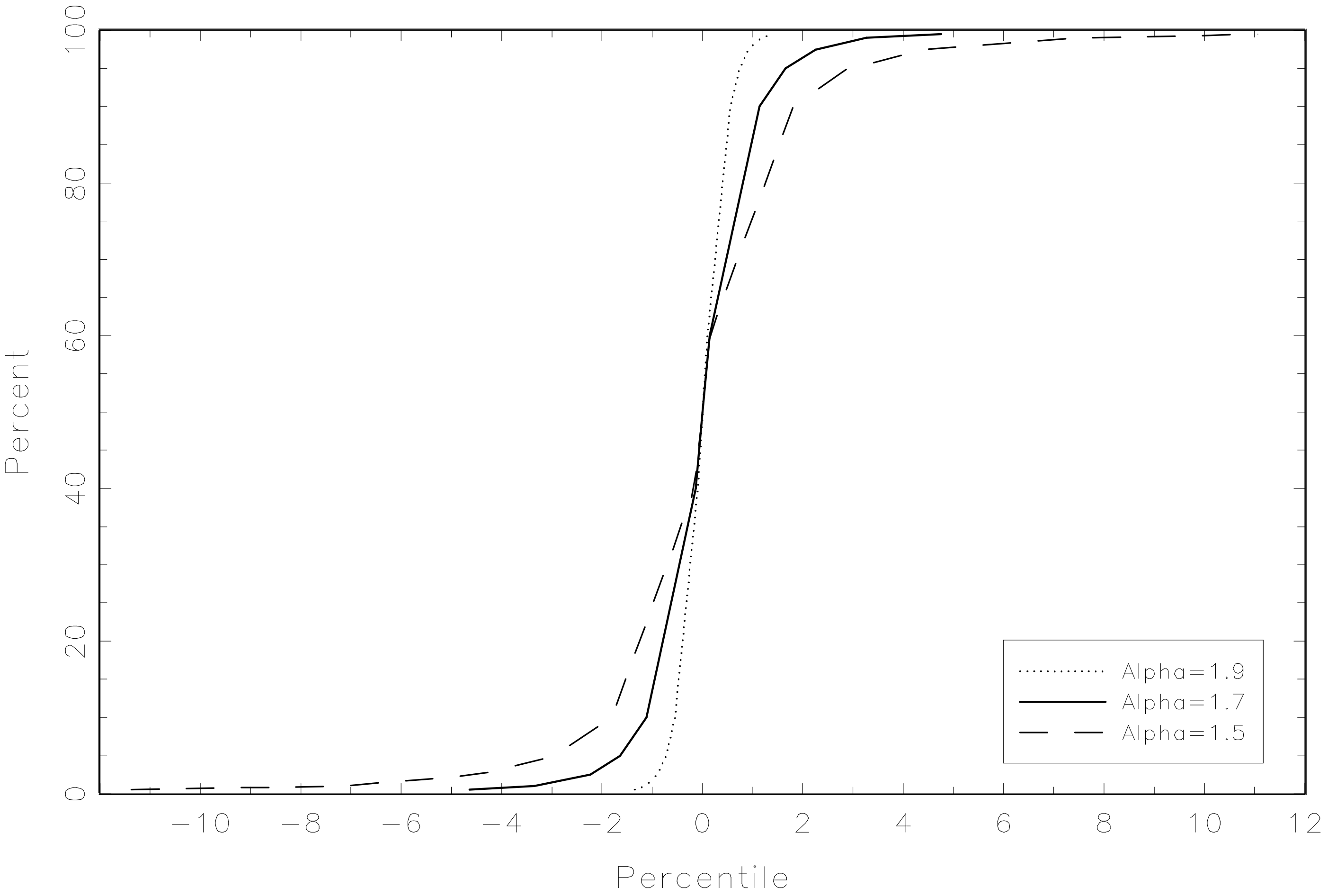


Figure 2: Asmptotic Distribution of the OVR Statistic: q=4

Figure 3: Asmptotic Distribution of the OVR Statistic: q=6

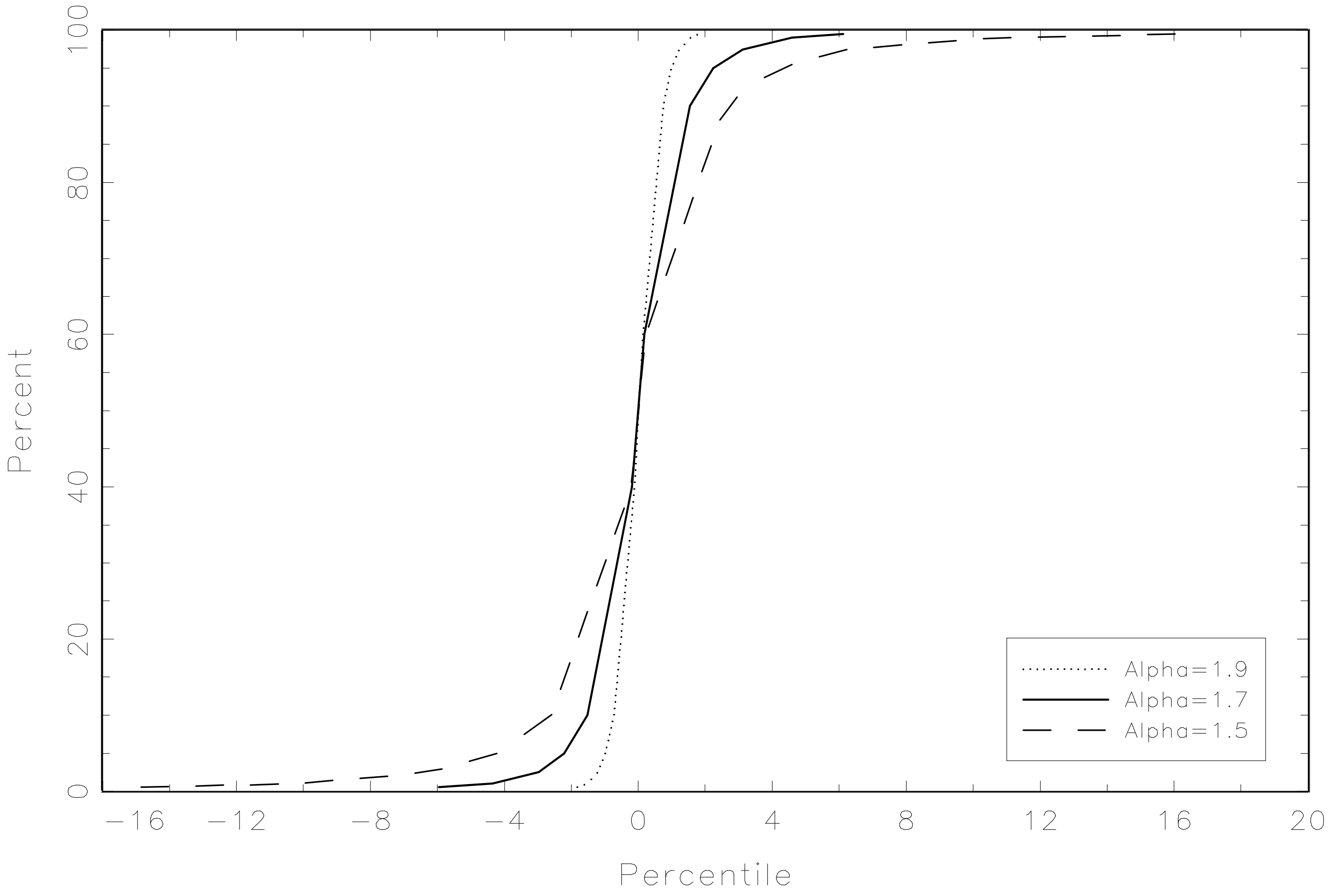


Figure 4: EQF versus Asmptotic Distribution of the OVR Statistic

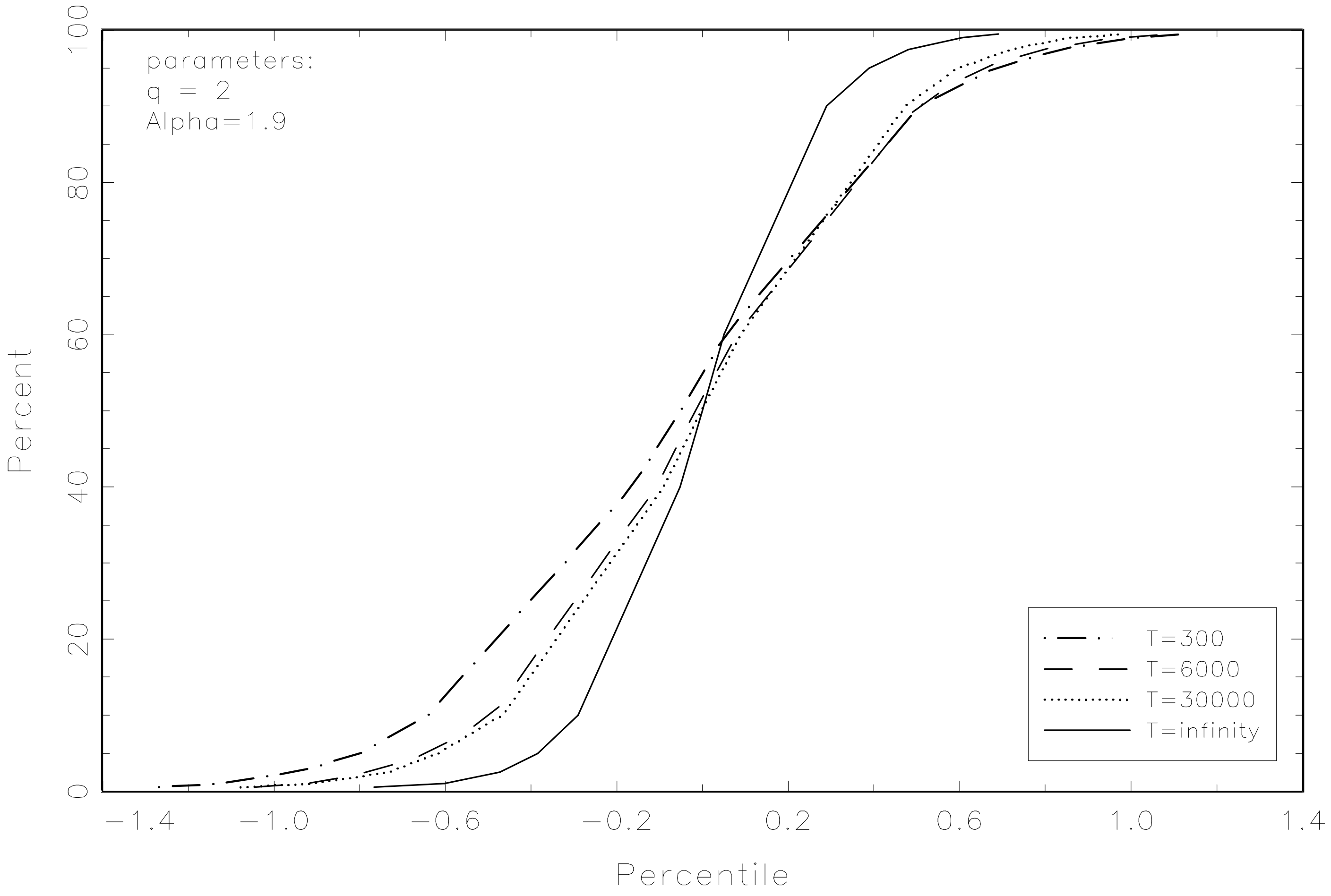


Figure 5: EQF versus Asmptotic Distribution of the OVR Statistic

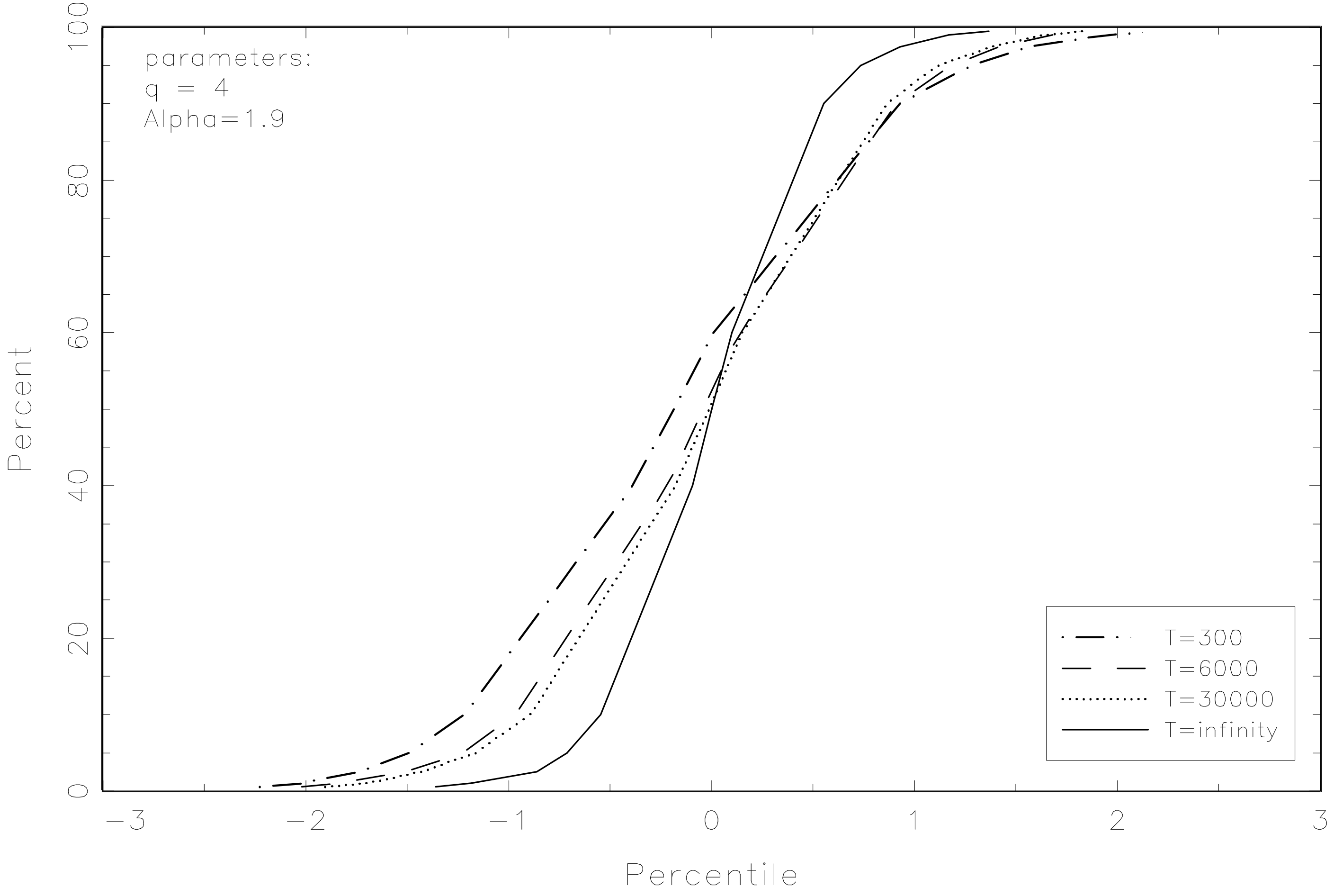


Figure 6: EQF versus Asmptotic Distribution of the OVR Statistic

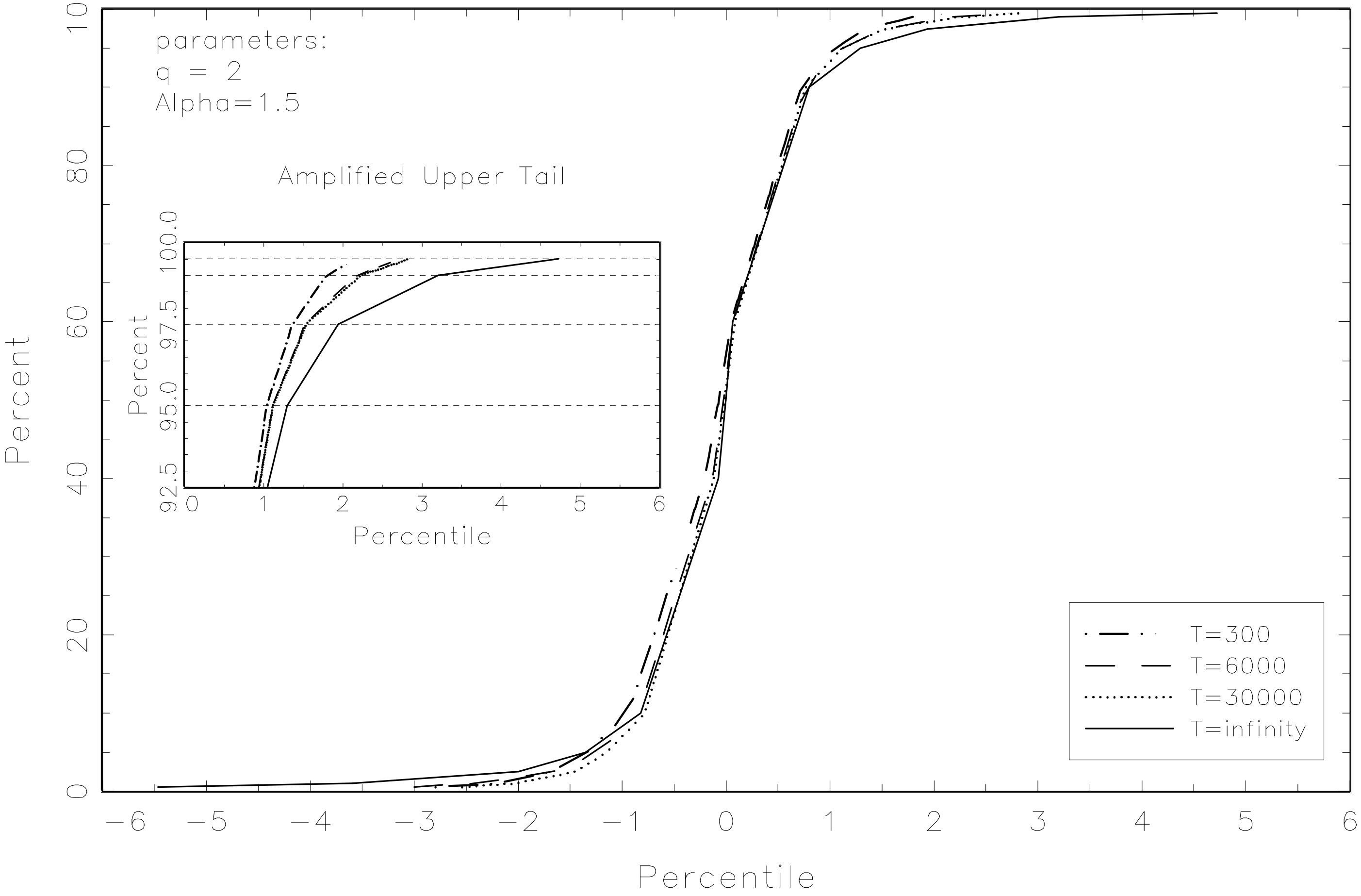


Figure 7: EQF versus Asmptotic Distribution of the OVR Statistic

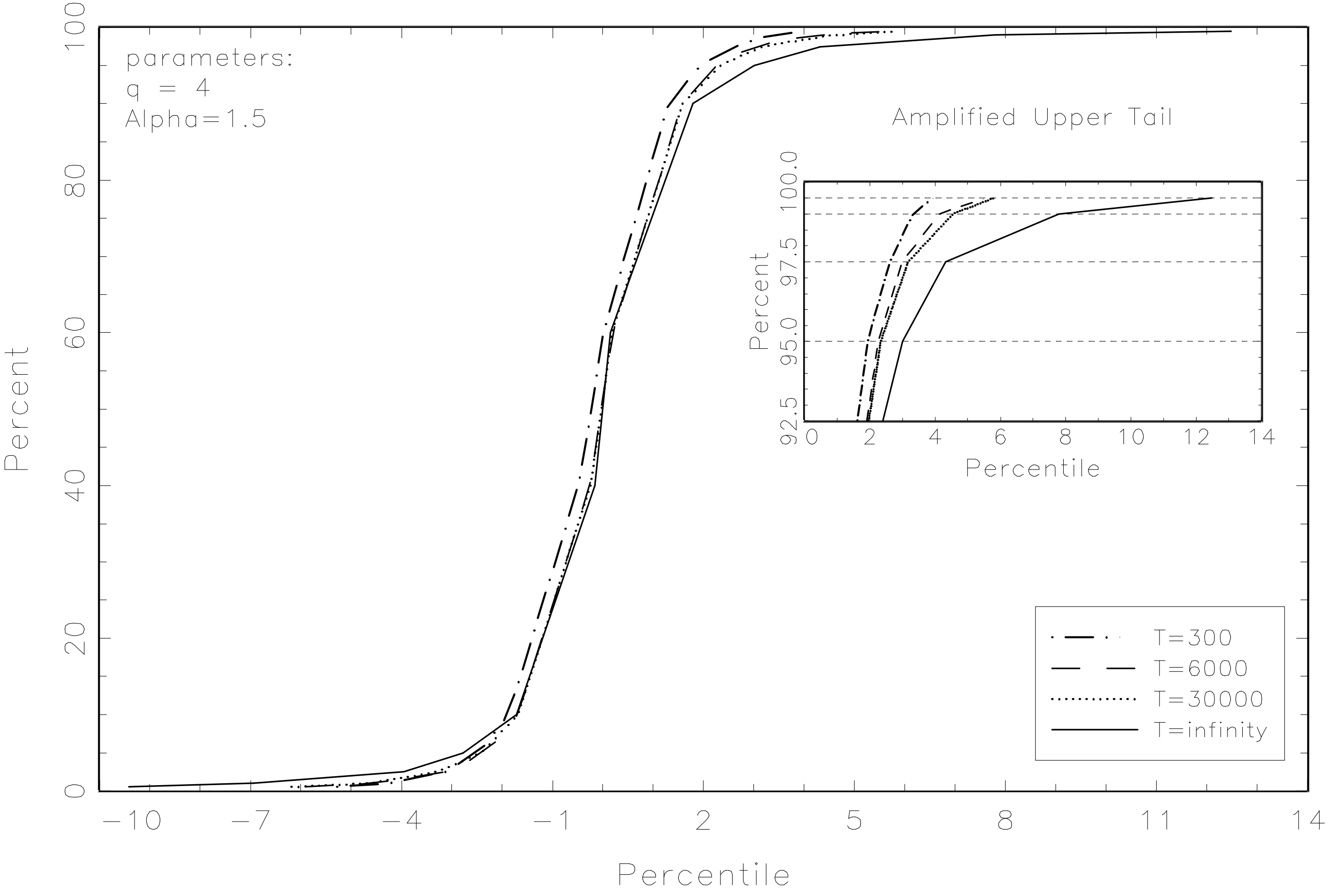


Figure 8: EQF versus Asmptotic Distribution of the OVR Statistic

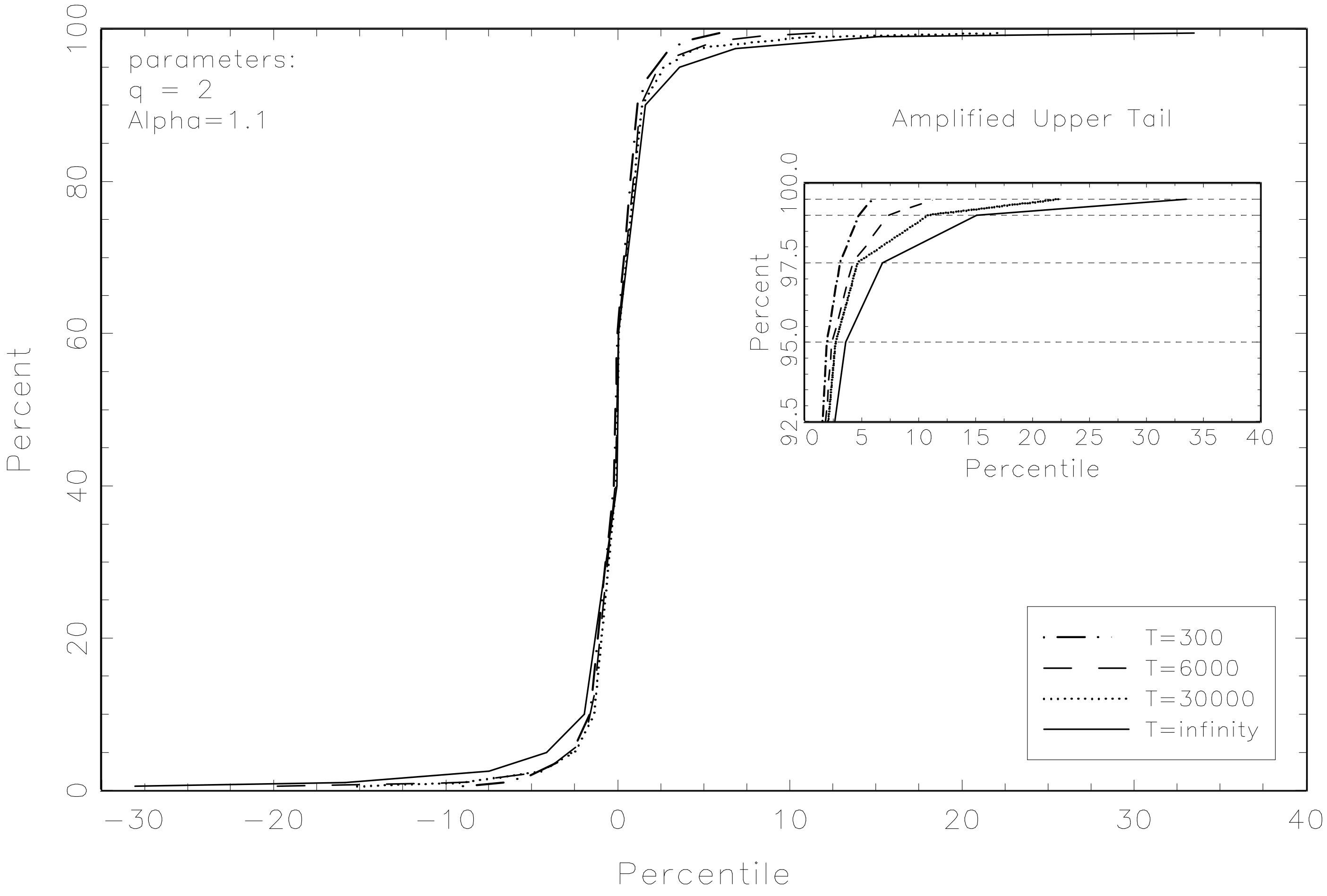


Figure 9: EQF versus Asmptotic Distribution of the OVR Statistic